On the Dynamics of the Demographic Dividend

A Formal Analysis of the Timing and Magnitude

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Abstract

When fertility declines, there is a transitory period of low dependency. This occurs when the largest cohorts born during the transition are of prime working ages and there are at once few children and few Elderly. The benefits arising from such an advantageous age structure are subsumed in the concept of the first demographic dividend.

The goal of this paper is to provide a formal framework for understanding the timing, duration, and magnitude of the gain from the "demographic window of opportunity" that accompanies the decline in fertility. We show that classical stable population theory is not able to quantify the magnitude of the effect arising from an advantageous age structure. Nevertheless, Coale's model of constantly declining fertility is capable to quantify age structure effects that are comparable in magnitude and duration to those of real populations.

From empirical research we know that both changes in and the level of the support ratio have an influence on economic growth (Kotschy et al., 2020). We focus on the level because it is the current level that determines the advantage in terms of output per capita. Moreover, this approach allows for a coherent quantification of the gain accumulated over the favourable period and an intuitive economic interpretation. Our formal demographic model explains how, when, how long and to what extent the fertility decline generates benefits in terms of labour supply. In addition, our approach allows for an accurate sensitivity analysis and projections of these benefits.

1 Introduction

One hopeful aspect of demographic change is the so-called "demographic window of opportunity" a transitory period of favourable age structure that occurs as fertility declines. After fertility has begun to decline, there is a period when there are still many workers, but few children and few elderly. As a result, per-capita income increases, both consumption and savings can increase, investments can be made in future generations, and institutions can be built to prepare a society and economy for the population aging that follows this transitory "boom." The demographic dividend is given some of the credit for Japan's go-go years of the 1980s and more recently for the economic rise of Korea and China. As Bloom and others emphasize, the demographic dividend is something to take advantage of when it occurs and as long as it lasts.

Our goal in this paper is to provide a formal analysis of the demographic dividend in terms of timing, duration, and magnitude. We begin with a review of stable population theory and what it tells us about the relationship between population growth, age-structure, and dependency. We will see that there is a growth rate r that minimizes dependency. The stable case, however, cannot describe the transient dynamics that we see in real populations, because the stability of fertility constrains the relationships between age groups, and does not allow a "bulge" in the working ages. We therefore turn next to a dynamic model of demographic change, first introduced by Coale (1972, ch. 4), in which fertility is forever declining. This model is implausible for the very distant past and very distant future but provides a useful model of what happens soon before and soon after fertility falls below replacement levels. We will see that the same framework used for understanding the minimization of dependency in stationary populations can also be used for so-called "pseudo-stable" populations described by Coale's model.

Many of the results we present here are based on findings in the book–length detailed analysis of Coale's model by Feichtinger and Vogelsang (1978). Our contribution here is to apply them to the problem of the demographic dividend.

2 A motivating example

The rapid fertility decline in China has created a temporary situation where the fraction of the population that is of working age is unusually high. Here we show UN estimates and forecasts of the support ratio in China from 1950 to 2100 and compare them with estimates from stable and pseudo-stable theory.

The first panel in the Figure 1 shows the support ratio for China as estimated and projected by the United Nations Population Division. We see that the support ratio decreased in the 1950s as improvements in child survival meant a larger share

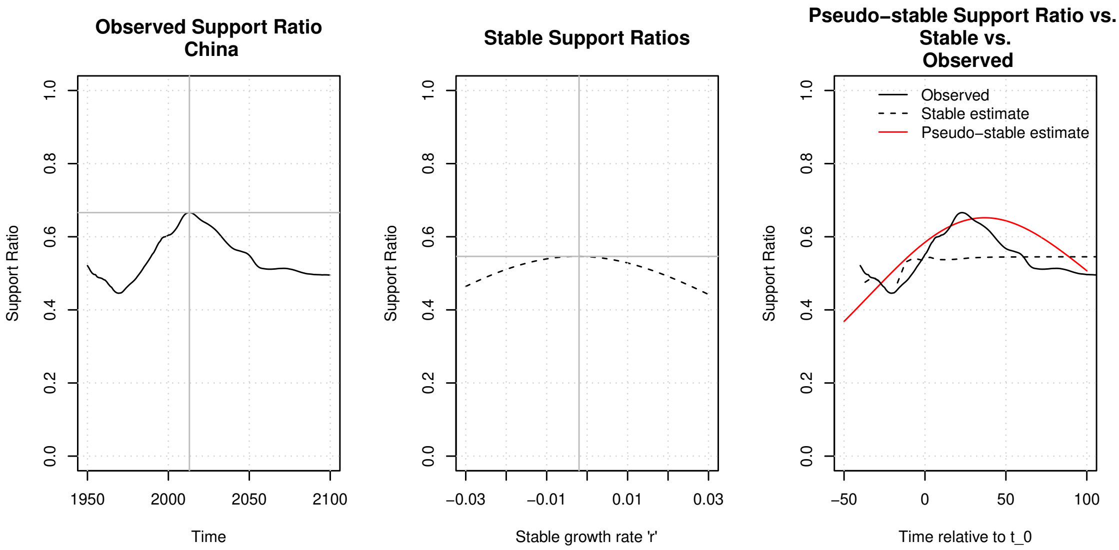

Figure 1: Support ratios in China estimated and projected by the United Nations compared to the possible range of support ratios in stable populations and the fitted pseudo-stable model. Panel A shows the support ratio estimated and projected by the United Nations, with the grey lines indicating the maximum level and timing of the maximum. Panel B shows the variation of the support ratio in a stable population across a broad range of growth rates using the Swedish period life table of 2000 as a model survival schedule. Panel C shows the observed (and projected) support ratio of China, with the time axis defined relative to the time when the period fertility rate reached replacemend t_0. The figure also shows the support ratio of the stable populations that would be obtained from the contemporary growth rates and the support ratio of the pseudo-stable population with fertility declining at a constant rate equal to that observed in China from $t_0 - 30$ to t_0 (2.4 percent). The stable population model is incapable of producing the observed variation in the support ratio, whereas the pseudo-stable population model provides a reasonable description in terms of both magnitude and timing.

of children in the population. Beginning in about 1970, as fertility began to fall, the support ratio began improving, reaching a maximum of $2/3$, or 1 dependent for every 2 workers in about 2015. After 2015, the support ratio is projected to decrease, as a result both of the ageing of the largest cohorts, and longevity improvements that increase the number of surviving elderly. By about 2060 or so, the sharp decrease in the support ratio is expect to come to an end and settle at levels similar to those observed before the onset of fertility decline.

The 2nd panel shows the variation in support ratios predicted from stable population theory. When populations are growing very quickly, the age pyramid will have so many children that the support ratio will be low. On the other hand, populations that grow very slowly, will have a large number of elderly. As the figure shows, in–between population growth rates, in this case, one just slightly negative, will maximise the long–term support ratio. The x–axes in panels 1 and 2 represent different kinds of variation, the observed support ratio is plotted against time, but the stable support ratio is plotted against the long–term growth rate. However, the y–axes are comparable, and show us that the highest support ratio in stable populations (about 0.6 in this case) is not nearly as high as observed in the real world case of China (about $2/3$).

The 3rd panel shows the support ratio implied by the pseudo–stable model of constantly declining fertility. The x–axis in this case is relative to the year in which fertility reaches replacement. The pseudo–stable model (which we will describe in detail further below) is able to account for several of the features of the real–world decline. It increases roughly in unison with the observed populations from the time when births peak to close to the maximum. The maximum support ratio in the pseudo–stable case ($2/3$) is now quite close to the maximum observed in China. The main difference is that the pseudo–stable case increases for somewhat longer than the projection by the UN. This is largely because the model includes only fertility change, and does not incorporate the increases in longevity – and accompanying increases in the non–working elderly – that are forecast to occur.

China is an example of very rapid fertility decline. It took only 2 decades for fertility to fall from an NRR of 2.0 to replacement levels. There are other countries – for example Indonesia – where the decline is expected to take more than twice as long. Our goal for the rest of this paper is to use a generalisable mathematical framework that will apply to all countries, and help explain what determines the timing, duration and magnitude of the demographic transition.

This specific example motivates us to use the pseudo–stable model to try to answer questions about the general features of the demographic window of opportunity and its dependence on the speed of fertility decline.

Namely,

1. When does the demographic dividend begin? Does it happen before or after

fertility declines to replacement?

2. How long does the dividend last? Does faster fertility decline lead to a briefer period of advantageous age structure?
3. How does the speed of fertility decline influence the height of the peak of the dividend, when age structure is at its most advantageous?
4. How can we measure the total size of the dividend, summing the benefits from each year? How does this magnitude vary with the speed of fertility decline?

3 The model

For a mathematical investigation of the dynamics of the support ratio during a decline in fertility we revisit a model proposed by Coale (1972, chap. 4). The basic assumption of this model is a population where fertility is fixed in age structure but declines at a constant rate $k < 0$. The mortality schedule is fixed and the survival function $l(a)$ denotes the probability of surviving to age a.

Coale (1972) derives from the mean value theorem of integral calculus that for any t there exists a $\mu(t)$ within the boundaries of reproductive age such that the number of births at time t is equal to the number of births $\mu(t)$ years ago (i.e. one generation before) times the net reproduction rate at time t,

$$B(t) = B(t - \mu(t))NRR(t).$$

This $\mu(t)$ is close to the mean age at childbearing. For mathematical convenience, we approximate $\mu(t)$ by a constant value μ —which we call henceforth generation length.

Although the aforementioned assumptions seem to be rather restrictive, there are indeed countries that exhibit a decline in the net reproduction rate that coincides with an exponential decline pretty well. Figure 2 illustrates the process of fertility decline for Brazil, China, Colombia, Costa Rica, Cuba, Pakistan and Republic of Korea from 1950 to 2021. The graphs show the actual data (dots) and the exponential decline according to our model (solid line). The time series of Brazil, Colombia and Costa Rica fit exceptionally well for a time period of more than half a century.

The respective rates of fertility decline, k, and the start and end of the fertility decline which we approximate by an exponential decline with rate k are listed in table 1. (If the end is 2021 this actually means that fertility decline is still going on this country but we used this last available data point to calculate k.)

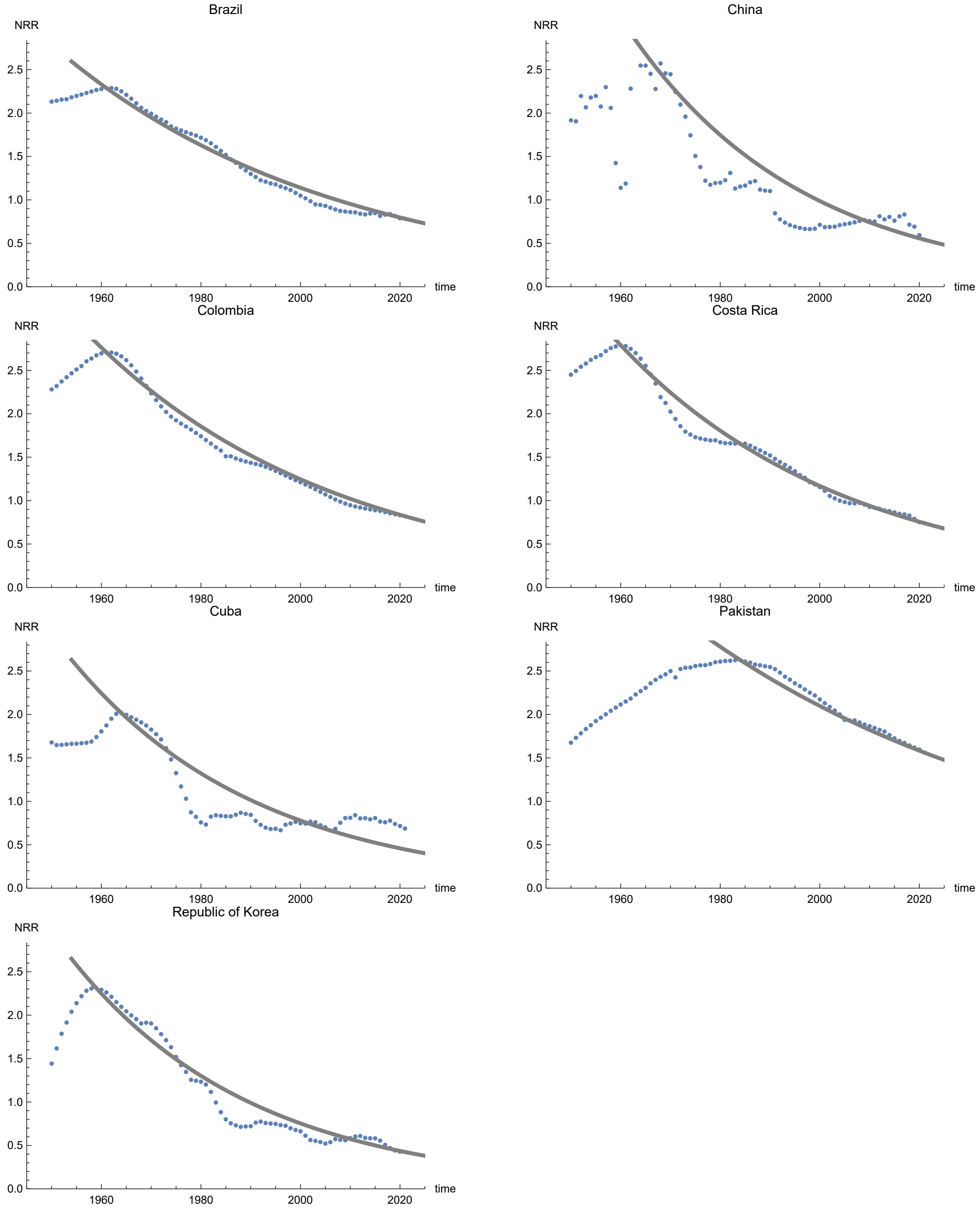

Figure 2: Decline of net reproduction rate in Brazil, China, Colombia, Costa Rica, Cuba, Pakistan and Korea from 1950 to 2021. Dots represent empirical data, solid lines illustrate exponential decline. Data source: United Nations, Department of Economic and Social Affairs, Population Division (2022).

country	k	start	end
Brazil	-0.018	1961	2021
China	-0.029	1963	2021
Colombia	-0.020	1961	2021
Costa Rica	-0.022	1960	2021
Cuba	-0.026	1964	2006
Pakistan	-0.014	1984	2021
Republic of Korea	-0.027	1959	2021

Table 1: Rates of fertility decline.

Regarding the assumption to keep generation length $\mu(t)$ constant at μ we have a look at the change of the mean age at childbearing in the same sample of countries over the same period of time (figure 3). The graphs show that in Costa Rica, Cuba and Pakistan the mean age at childbearing changed very little during the time interval from 1950 to 2021. Thus, Costa Rica is the only country in this example that fulfils both assumption, an exponential decline of the net reproduction rate and an almost constant generation length.

To analyze the model, we normalize the time scale such that $t = 0$ when the net reproduction rate $NRR = 1$. In this formal model, fertility approaches infinity for $t \to -\infty$ and zero for $t \to \infty$. Although these extreme values are unrealistic, the model is appropriate for describing the development of the age structure during the transition phase from high to low levels of fertility. For the limiting case $k = 0$, we assume that the NRR is constantly equal to 1. This special case actually represents a stationary population but it is relevant for our analysis as it marks the boundaries for the dynamics in the case of a slow fertility decline.

To investigate the age structure, we denote $N(a,t)$ the number of females aged a at time t, which is equal to the number of births at time $t-a$ times the survival probability $l(a)$, thus $N(a,t) = B(t-a)l(a)$. Then, the number of births at time t becomes (Coale, 1972, equ. (4.8))

$$B(t) = B(0)\exp\left(\frac{k}{2}t + \frac{k}{2\mu}t^2\right).$$

This results in the number of births at time $t-a$ as

$$B(t-a) = B(t)g(a,t)$$

with (see Coale (1972, p. 120) and Feichtinger and Vogelsang (1978, p. 22))

$$g(a,t) = \frac{B(t-a)}{B(t)} = \exp\left(-\frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a\right). \tag{1}$$

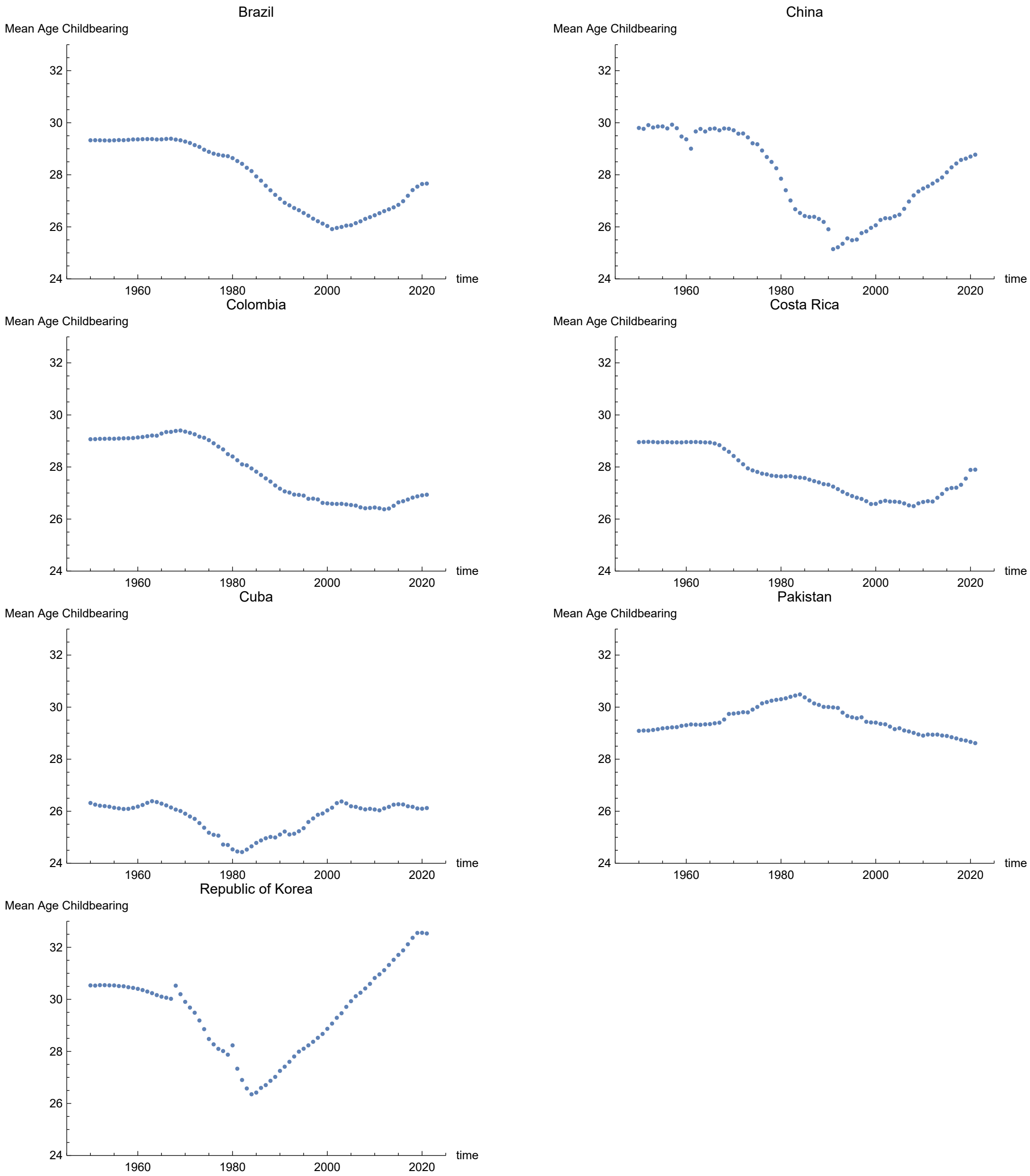

Figure 3: Mean age at childbearing in Brazil, China, Colombia, Costa Rica, Cuba, Pakistan and Korea. Data source: United Nations, Department of Economic and Social Affairs, Population Division (2022).

The derivation is given in appendix A.

With this we express the birth rate according to Coale (1972, equ. (4.38)) and Feichtinger and Vogelsang (1978, equ. (6.4)) as

$$b(t) = \frac{1}{\int_0^\omega g(a,t)l(a)da}, \tag{2}$$

where ω denotes the maximum age of the life table. The support ratio is defined as the share of the working age population in the total population, i.e

$$S(t) = \frac{\int_W N(a,t)da}{\int_0^\omega N(a,t)da} = \frac{\int_W g(a,t)l(a)da}{\int_0^\omega g(a,t)l(a)da}, \tag{3}$$

with W denoting the age interval of the working age population.

From empirical research we know that both, changes in the support ratio and the level of the support ratio have a significant influence on economic growth (Kotschy et al., 2020). Therefore, we investigate the time interval during which the support ratio $S(t)$ increases, and the time interval during which the support ratio exceeds a certain threshold. For this threshold we choose the support ratio of a stationary population S_0 with the same mortality schedule $l(a)$, which we express as

$$S_0 = \frac{\int_W l(a)da}{\int_0^\omega l(a)da}. \tag{4}$$

We choose this level as a benchmark as it is a neutral level that can be sustained with a constant age structure and consider a support ratio above S_0 advantageous. For a decrease in fertility, i.e. $k < 0$, the support ratio initially increases and then decreases. Therefore, we look for those points in time t_1 and t_2 when the support ratio of the pseudo–stable population $S(t)$ equals the support ratio of the stationary population S_0. During the time interval in between these two points in time, the support ratio of the pseudo–stable population $S(t)$ exceeds the support ratio of the stationary population S_0. Finally, we calculate the area between the support ratio of the pseudo–stable population $S(t)$ and the stationary population S_0 from time t_1 to time t_2. The size of this area informs us about the total amount of the surplus a population gains from a declining fertility.

4 Analysis

We start our analysis by deriving the time t^* when the support ratio (3) peaks. Using (Feichtinger and Vogelsang, 1978, equ. (6.92)) we write the dynamics of the support ratio as

$$\frac{d}{dt}\log S(t) = \frac{k}{\mu}\left[A(t) - A_W(t)\right] = \frac{\dot{S}(t)}{S(t)}, \tag{5}$$

where $A(t)$ and $A_W(t)$ denote the mean ages of the total and working age population, respectively. From that we conclude that the support ratio peaks when the mean age of the total population $A(t)$ equals the mean age of the working age population $A_W(t)$. Solving the equation $A(t) = A_W(t)$ for t we obtain

Proposition 1 *The time t^* when the support ratio reaches its maximum is approximately*

$$t^* = \frac{(A_0 - A_{W,0})\frac{\mu}{k} + t_0\sigma_0^2 - t_{W,0}\sigma_{W,0}^2}{\sigma_0^2 - \sigma_{W,0}^2} \tag{6}$$

with t_0 ($t_{W,0}$) denoting the time when the mean age of the total population $A(t)$ (working age population $A_W(t)$) is equal to the mean age of the stationary population A_0 (stationary working age population $A_{W,0}$) with the same survival function $l(a)$. In the special case $A_0 = A_{W,0}$ and if lifespan inequality is sufficiently low this can be further approximated as

$$t^* = A_0 - \frac{\mu}{2}. \tag{7}$$

The proof is given in appendix A. The subscript 0 indicates variables that refer to a stationary population based on the same life table as the population currently investigated. Thus, A_0 and $A_{W,0}$ are the mean ages of the corresponding stationary population and stationary working age population and σ_0^2 and $\sigma_{W,0}^2$ denote the variance in age of the corresponding stationary total population and stationary working age population, respectively.

From (7) we conclude that from time $t = 0$ when the net reproduction rate is equal to one it takes approximately the mean age of the corresponding stationary population minus half a generation length to arrive at the maximum support ratio. Moreover, it follows from approximations (7) and (30) that the maximum support ratio is reached at about the same time that the mean age of the pseudo–stable population is equal to the mean age of the stationary population.

To examine the sensitivity of t^* to changes in k and μ we compute the derivatives

$$\frac{dt^*}{dk} = -\frac{A_0 - A_{W,0}}{\sigma_0^2 - \sigma_{W,0}^2}\frac{\mu}{k^2} \tag{8}$$

$$\frac{dt^*}{d\mu} = \frac{A_0 - A_{W,0}}{\sigma_0^2 - \sigma_{W,0}^2}\frac{1}{k} - \frac{1}{2}. \tag{9}$$

The difference of the variances $\sigma_0^2 - \sigma_{W,0}^2$ and the fraction μ/k^2 are always positive. The difference of the mean ages $A_0 - A_{W,0}$ depends on the survival schedule and on the age limits of the working age population. In general this difference is positive in highly developed regions with high life expectancy but may be negative in less and least developed regions with low life expectancy. If $A_0 = A_{W,0}$ we get ${dt^*}/{dk} = 0$, which means that the peak time t^* does not depend on the speed of fertility decline k. In the case of $A_0 > A_{W,0}$, the derivative dt^*/dk is negative, thus an increase in k (since k is negative this means a slowdown of fertility decline) causes an acceleration of the peak of the support ratio. In the case of $A_0 < A_{W,0}$, an increase in k results in an delay of the peak. The derivative ${dt^*}/{d\mu}$ is zero (which means t^* does not depend on μ) if $A_0 - A_{W,0} = (\sigma_0^2 - \sigma_{W,0}^2){k}/{2}$. An increase in μ delays (accelerates) the peak time of the support ratio if $A_0 - A_{W,0} > (\sigma_0^2 - \sigma_{W,0}^2){k}/{2}$ $(A_0 - A_{W,0} < (\sigma_0^2 - \sigma_{W,0}^2){k}/{2})$.

The support ratio grows as long as the time derivative in (5) is positive. Since $k < 0$ and $\mu > 0$ this is the case as long as $A(t) < A_W(t)$, i.e. the mean age of the total population deceeds the mean age of the working age population. In the framework of the pseudo–stable population model we start with a very young population. Therefore, the youth dependency is high and $A(t) < A_W(t)$. Since fertility decreases monotonically, this inequality is fulfilled until t^* given in (6) when $A(t) = A_W(t)$. Thereafter the increase in the old age dependency depresses the support ratio resulting in $A(t) > A_W(t)$. We conclude that the share of the working age population, i.e. the support ratio, increases for all $t < t^*$.

In the next step we look at the maximum surplus in terms of the support ratio that can be achieved due to a decline in fertility. Therefore, we evaluate the support ratio $S(t)$ at t^* and calculate the difference to S_0, the support ratio of the corresponding stationary population. Henceforth, we call this difference the height h. For this we introduce the ith moments L_i for the total population (Keyfitz, 1985, p. 89) and L_i^W for the working age population,

$$L_i = \int_0^{\omega} a^i l(a) da \quad \text{and} \quad L_i^W = \int_W a^i l(a) da. \tag{10}$$

to arrive at

Proposition 2 *The maximum surplus of the support ratio is approximately*

$$\begin{aligned} h &= S(t^*) - S_0 \\ &= \frac{8L_0^W \mu^2 + 4k\mu\left[L_2^W - L_1^W(2t^* + \mu)\right] + k^2(2t^* + \mu)(L_2^W(2t^* + \mu) - 2L_3^W)}{8L_0\mu^2 + 4k\mu\left[L_2 - L_1(2t^* + \mu)\right] + k^2(2t^* + \mu)(L_2(2t^* + \mu) - 2L_3)} - \frac{L_0^W}{L_0}. \end{aligned} \tag{11}$$

Again we look at the special case $A_0 = A_{W,0}$ and assume that lifespan inequality is *sufficiently to obtain the approximation*

$$
\begin{aligned}
h = & \Big\{ k \left[A_{W,0} k (A_{W,0} L_0 L_2^W - A_{W,0} L_2 L_0^W + L_3 L_0^W - L_0 L_3^W) \right. \\
& \left. + (2A_{W,0} L_1 L_0^W - 2A_{W,0} L_0 L_1^W - L_2 L_0^W + L_0 L_2^W)\mu \right] \Big\} \Big/ \\
& \Big\{ L_0 \left[A_{W,0} k^2 (A_{W,0} L_2 - L_3) + k(L_2 - 2A_{W,0} L_1)\mu + 2L_0 \mu^2 \right] \Big\} .
\end{aligned} \tag{12}
$$

Due to the length and complexity of (11) it does not allow for a straightforward interpretation but from (12) we conclude that the height h is approximately proportional to the speed of fertility decline expressed as $|k|$ and approximately indirectly proportional to the generation length μ. We will see later on that this also holds true for the general case.

Finally, we investigate the intersections of the support ratio $S(t)$ of the pseudo–stable population and the support ratio S_0 of the corresponding stationary population. The time interval in between these intersections is the advantageaus period when an economy enjoys a surplus of labor supply that is above a level that could be sustained in the long run.

Proposition 3 *The points in time that mark the beginning and end of a support ratio $S(t)$ that exceeds S_0 are approximately*

$$
\begin{aligned}
t_{1,2} = & \, 2\mu L_0 L_0^W (A_0 - A_{W,0}) \pm \Big\{ (L_0^W L_3 - L_0 L_3^W)^2 k^2 - \Big[L_0^{W\,2} (L_2^2 - L_1 L_3) \\
& + L_0 L_0^W (-2L_2 L_2^W + L_1^W L_3 + L_1 L_3^W) + L_0^2 (L_2^{W\,2} - L_1^W L_3 W) \Big] 4\mu k \\
& + \left[-L_0 L_3^W + L_0 L2^W \mu + L_0^W (L_3 - L_2 \mu) \right] k \Big\}^{1/2} \Big/ \left[(L_0^W L_2 - L_0 L_2^W) 2k \right] .
\end{aligned} \tag{13}
$$

The center of t_1 and t_2 does not depend on k but its sensitivity with respect to generation length μ depends on whether $A_0 > A_{W,0}$ or $A_0 < A_{W,0}$. The length (duration) of the advantageous support ratio, i.e. $t_2 - t_1$ is approximately indirectly proportional to $\sqrt{-k}$ and approximately proportional to $\sqrt{\mu}$.

To compute the total amount of the surplus gained from a declining fertility we calculate numerically the Riemann sum of $S(t) - S_0$ over the time interval from t_1 to t_2 and call the result A_{RS}. To obtain an analytical expression, we approximate the area between the two curves by the area of a parabola with equal height and length, i.e.

$$A_P = {}^2\!/_3\, h(t_2 - t_1) \,. \tag{14}$$

5 Results

To illustrate the dynamics of the model described and analysed in sections 3 and 4 and to investigate the model's sensitivity with respect to changes in parameters we start with a stylised mortality schedule and afterwards present results based on empirical life table data from United Nations, Department of Economic and Social Affairs, Population Division (2019).

We start our analysis with a survival function where all mortality is concentrated at one age. This means everyone survives to age ω and there is no lifespan inequality,

$$l(a) = \begin{cases} 1 & if\, a < \omega \\ 0 & if\, a = \omega \end{cases} . \tag{15}$$

The peak support ratio occurs when $A(t) = A_W(t)$, i.e the mean ages of the total population and the working age population are equal. The point in time at which this equality occurs depends on the working age interval W. We assume that working age starts at $\underline{w}$ and ends at $\overline{w}$ with $0 \leq \underline{w} \leq \overline{w} \leq \omega$ which implies $[\underline{w}, \overline{w}] \subseteq [0, \omega]$.

Proposition 4 *In the case of concentrated mortality, the time t^* when the support ratio peaks is approximately*

$$t^* = \frac{(\underline{w} + \overline{w})^2 k t_{W,0} + 6\mu(\underline{w} + \overline{w} + \omega) - k t_0 \omega^2}{k[(\underline{w} - \overline{w})^2 - \omega^2]}. \tag{16}$$

In the special case $A_0 = A_{W,0}$ which means that the working age interval W is symmetric around $\omega/2$ we get

$$t^* = \frac{\omega - \mu}{2}. \tag{17}$$

The solution in (17) is equivalent to (7) since in the case of concentrated mortality the mean age of the corresponding stationary population becomes $A_0 = \omega/2$.

To obtain concrete results we investigate the special case $\underline{w} = \omega/4$ and $\overline{w} = 3\omega/4$. We get the maximum support ratio, if we evaluate $S(t)$ at t^*, which is approximately

$$S(t^*) = \frac{203}{256} + \frac{45\mu(3k\omega^2 - 100\mu)}{64(240\mu^2 - 20k\mu\omega^2 + k^2\omega^4)}, \tag{18}$$

and its derivative with respect to k becomes

$$\frac{dS(t^*)}{dk} = -\frac{45\mu\omega^2(1280\mu^2 - 200k\mu\omega^2 + 3k^2\omega^4)}{64(240\mu^2 - 20k\mu\omega^2 + k^2\omega^4)^2}. \tag{19}$$

Since $k < 0$ and $\mu, \omega > 0$ the derivative $dS(t^*)/dk$ in (19) is negative. Thus, if the absolute value of k increases (i.e. k decreases), which means a more rapid decline of fertility, the peak support ratio becomes higher — as expected. To derive the maximum surplus we subtract S_0, the support ratio of the stationary population, to obtain

$$h = \frac{75}{256} + \frac{45\mu(3k\omega^2 - 100\mu)}{64(240\mu^2 - 20k\mu\omega^2 + k^2\omega^4)}. \tag{20}$$

The beginning and end of the time period when $S(t) > S_0$ are approximately

$$t_{1,2} = -\frac{\mu}{2} + \frac{3\omega}{4} \pm \frac{\sqrt{9k^2\omega^2 - 64k\mu}}{8k}. \tag{21}$$

As in the general case (13), the center does not depend on k but an increase in μ shifts the center to the left. The surplus lasts for a time period with length

$$l = -\frac{\sqrt{9k^2\omega^2 - 64k\mu}}{4k} \tag{22}$$

and is approximately proportional to $\sqrt{\mu}$ and $1/\sqrt{-k}$. The total amount of the surplus, approximated by the area under a parabola, is

$$A_P = -\frac{5\sqrt{9k^2\omega^2 - 64k\mu}\,(5k\omega^4 - 64\mu\omega^2)}{512(240\mu^2 - 20k\mu\omega^2 + k^2\omega^4)}. \tag{23}$$

For $-1 \ll k < 0$ it follows that $k^2 \ll |k|$. Thus, the numerator grows approximately proportionally to $|k|^{3/2}$ and the denominator grows approximately proportionally to $|k|$. Therefore, the total amount is approximately proportional to $\sqrt{-k}$. The first order approximations of the height h, the reciprocal of the length $1/l$ and of the total amount A_P with respect to k about $k = 0$ for $k < 0$ are

$$\begin{aligned} h &= -\frac{\omega^2 k}{64\mu} + O(k^2) \\ 1/l &= \frac{1}{2}\sqrt{\frac{-k}{\mu}} + O(k^{3/2}) \\ A_P &= \frac{\omega^2\sqrt{-k}}{48\sqrt{\mu}} + O(k^{3/2}). \end{aligned} \tag{24}$$

From this we conclude that for small $|k|$ the height h is approximately proportional to $|k|$, the length l is approximately indirectly proportional to $\sqrt{-k}$ and the total amount A_P is approximately proportional to $\sqrt{-k}$.

In figure 8 we illustrate the sensitivity of the time t^* when the support ratio peaks with respect to changes in the speed of fertility decline k and the generation length μ and the accuracy of the approximations (7) and (17). The red lines represent the values of t^* in the case of concentrated mortality. In the left panel μ is fixed at 30 and the speed of fertility decline k ranges from -0.1 to -0.002. In the right panel k is fixed at -0.01 and μ varies from 15 to 45. The solid red lines depict the results obtained from numerical evaluations and the dashed red lines result from the approximation (17). We see that changes in k do not affect t^* but it declines with increases in μ as expected from (17). Since the approximation is perfectly accurate in the case of concentrated mortality and $A_0 = A_{W,0}$, the dashed and solid red lines coincide.

Figure 4 depicts the support ratio over time for different levels of k, the speed of fertility decline. All curves are calculated using (1), generation length $\mu = 30$ and maximum life span $\omega = 80$. The time scale is chosen according to the Coale model, i.e. $time = 0$ when $NRR = 1$. As expected from (17), the graphs confirm that t^*, the point in time when the support ratio reaches its maximum value, does not depend on k. The trajectory with the slowest fertility decline, i.e. $k = -0.005$, reaches the intersection with the support ratio of the stationary population S_0 first and stays above that level for the longest time span. Moreover, the ascending and descending branch of the support ratio are symmetric with respect to $t^* = {}^{(\omega-\mu)}/_2$. This is in contradiction to (21) indicating a midpoint at ${}^{3\omega}/_4 - {}^{\mu}/_2$. This deviation results from the approximations used to derive (21) which delays the intersections t_1 and t_2.

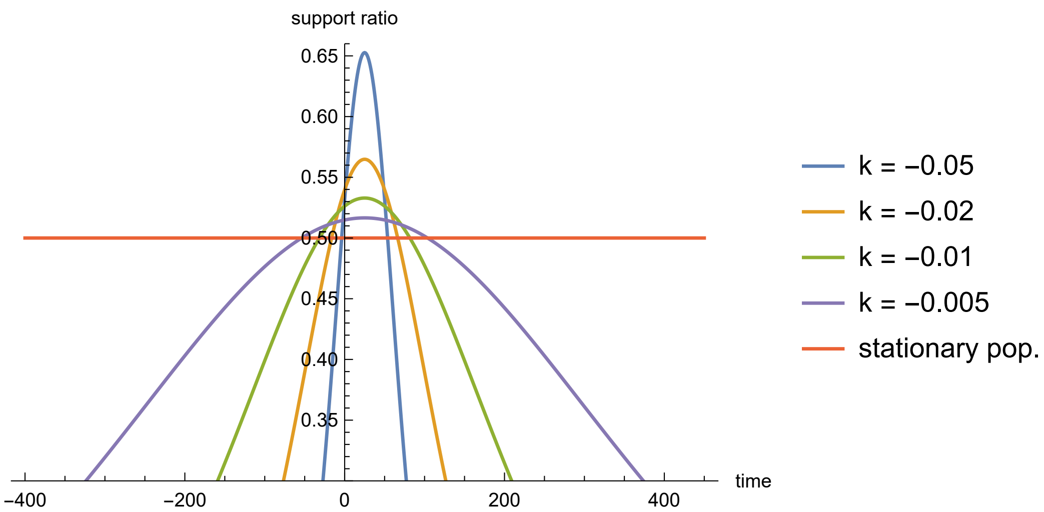

Figure 4: Support ratio over time, concentrated mortality, $\mu = 30$, $\omega = 80$, $time = 0$ when $NRR = 1$.

Figure 5 shows the same trajectories as figure 4 but the time scale is shifted such that $time = 0$ when $NRR = 2$. As expected, the higher the speed of fertility decline, the earlier the population arrives at t^*. Moreover, also the intersections with the support ratio of the stationary population S_0 occurs earlier if fertility declines faster. Therefore, starting at $NRR = 2$, the population arrives earlier in the stage which we consider the demographic window of opportunity if fertility decline is faster.

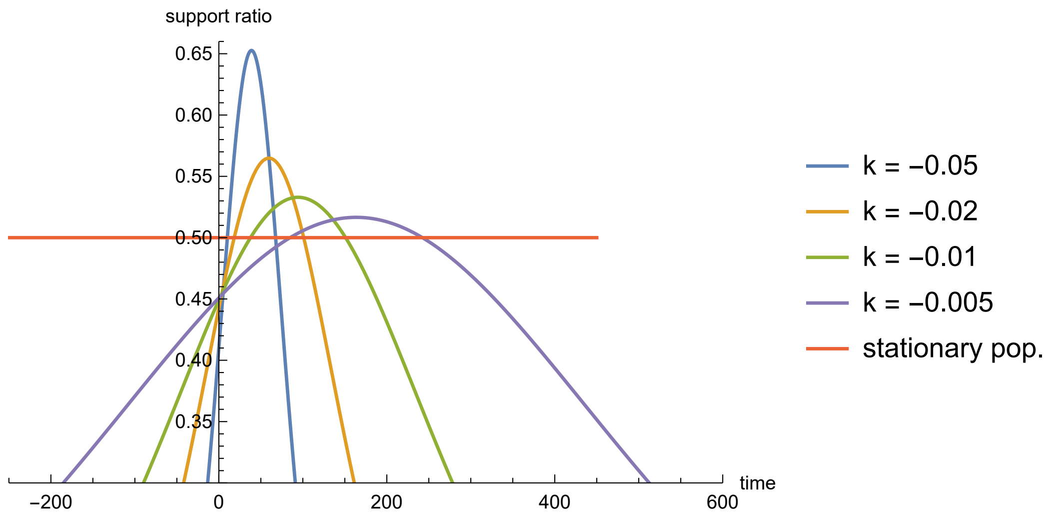

Figure 5: Support ratio over time, concentrated mortaility, $\mu = 30$, $\omega = 80$, $time = 0$ when $NRR = 2$.

The sensitivity of the total amount A, height h and lenght l of the surplus in the support ratio with respect to changes in k is given figure 10. Again the solid red lines represent the results for the case of concentrated mortality. The first graph in the left column depicts the total amount vs. k. It declines along a concave path which is approximately proportional to $\sqrt{-k}$ as we expected from (24). To illustrate this proportionality more clearly, the first graph in the right column shows $A/\sqrt{-k}$ vs. k. Since this line is almost horizontal, the total amount A is indeed proportional to $\sqrt{-k}$. The second graph in the left column shows the height h, i.e. the difference between the peak support ratio and the support ratio of a stationary population with the same survival schedule, vs. k. Since the decline is almost linear and because of our findings from (24) we expect that h is proportional to $-k$ and plot $h/-k$ in the second graph of the right column. The red line increases with a moderate slope indicating that the proportionality is not as accurate as it was in the case of the total amount A. The last graph in the left column shows the length l vs. k. Again because of (24), we expect that the length is indirectly proportional to $\sqrt{-k}$ and plot $l\sqrt{-k}$ in the last graph on the right hand side. Again the line has a slope which indicates some deviations from the expected proportionality due to higher order terms.

Fig. 6 illustrates the dynamics of the support ratio over time if the generation length μ takes on the values 20, 30 and 40. The curves are calculated with a speed of fertility decline $k = -0.01$ and the function for $g(a,t)$ given in (1). An increase in μ shifts the center to the left confirming (17) and flattens the curve.

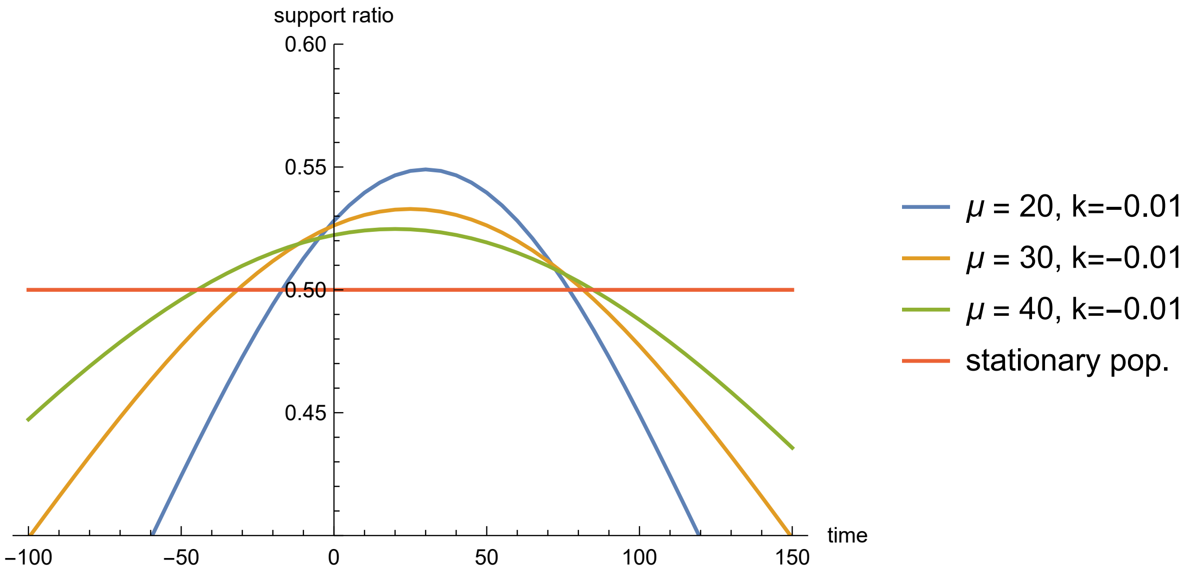

Figure 6: Support ratio, concentrated mortality, $k = -0.01$, $\omega = 80$.

For a detailed analysis of the sensitivity of A, h and l with respect to μ we have a look at the solid red lines in figure 12 representing the results based on a concentrated mortality schedule. The left column illustrates A, h and l vs. μ and the right column shows $A/\sqrt{\mu}$, $h\mu$ and $l/\sqrt{\mu}$ vs. μ. We see that the total amount A and the height h decline but the length decreases with an increase of μ. In the right columns we again check to what extent these quantities are proportional to $\sqrt{\mu}$, $1/\mu$ and $\sqrt{\mu}$, respectively. The red lines in the second and third graph of the right panel are almost horizontal, indicating that the height h and the length l of the surplus in the support ratio are indeed approximately proportional to $1/\mu$ and $\sqrt{\mu}$. In the case of the total amount this is not the case.

In the following, we examine the dynamics of the support ratio using empirical life tables from United Nations, Department of Economic and Social Affairs, Population Division (2022). Since our model is in continuous age and time but life table data are provided for discrete age groups, we use the mortality model of

Makeham (1867) which uses three parameters α, β and γ,

$$l(a) = \exp\left[\frac{\alpha}{\beta}\left(1 - e^{\beta a}\right) - \gamma a\right]. \tag{25}$$

Since we know from section 4 that the sign of $A_0 - A_{W,0}$ is crucial for the sensitivity of several results, we choose life tables from three countries at three particular points in time when the mean ages of the stationary populations resulting from the given life tables satisfy $A_0 - A_{W,0} > 0$, $A_0 - A_{W,0} < 0$ and $A_0 - A_{W,0} \approx 0$. Table 2 depicts the respective values.

country	time	$A_0 - A_{W,0}$
Japan	2015-2020	1.082
Republic of Korea	1980-1985	-4.761
Sweden	2015-2020	0.063

Table 2: Differences of mean ages $A_0 - A_{W,0}$ for stationary populations resulting from empirical life tables.

Figure 7 illustrates the life table survivors for these three countries and the survival functions in continuous age resulting from fitting (25) to the life table data. The fit is very accurate except for the case of high child mortality. Since the

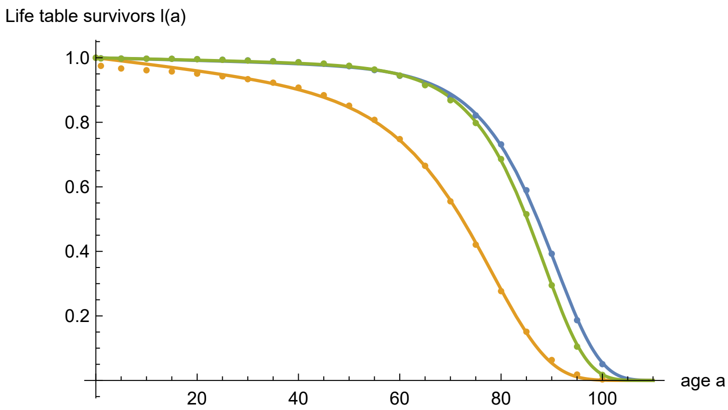

Figure 7: Life table survivors for Japan 2015-2020, Republic of Korea 1980-1985 and Sweden 2015-2020. The dots represent empirical data and the solid lines depict the survival functions resulting from fitting the Makeham model given in (25). Data source: United Nations, Department of Economic and Social Affairs, Population Division (2019).

focus of our analysis is on the dynamics of the share of the working age population, this three parameter model is sufficient for our purpose.

The sensitivity of t^* with respect to changes in the speed of fertility decline k and generation length μ for these three life tables is illustrated in figure 8. The left panel depicts t^* over k, the right panel shows t^* over μ. Solid lines represent the results from numerical evaluations and dashed lines were determined using approximation (7). We see that in the case of the Swedish life table the approximation is pretty accurate because the difference $A_0 - A_{W,0}$ — which we neglect when approximating (6) by (7) — is small. In the case of life tables from Japan and Korea the deviations from the approximations are much larger because $A_0 - A_{W,0} > 0$ for Japan and $A_0 - A_{W,0} < 0$ for Korea. The left panel shows that the deviation becomes larger for smaller absolute values of k and the right panel shows that it becomes larger for larger μ. This happens because in (6) the difference $A_0 - A_{W,0}$ is multiplied by the fraction μ/k. This fraction is negative since $\mu > 0$ and $k < 0$ and then divided by the difference $\sigma_0^2 - \sigma_{W,0}^2$, which is positive, because the age variance of the total population, σ_0^2, is greater than the age variance of the working age population, $\sigma_{W,0}^2$. As a consequence, in the case of Japan the exact value of t^* is greater than the approximation (7) for small absolute values of μ/k but smaller for large absolute values of μ/k. What is remarkable is that in the case of a very slowly declining fertility the maximum support ratio is already achieved before $NRR = 1$ at time $t = 0$. In the case of Korea the difference $A_0 - A_{W,0}$ has a larger absolute value and, in turn, the exact value of t^* always exceeds the approximation.

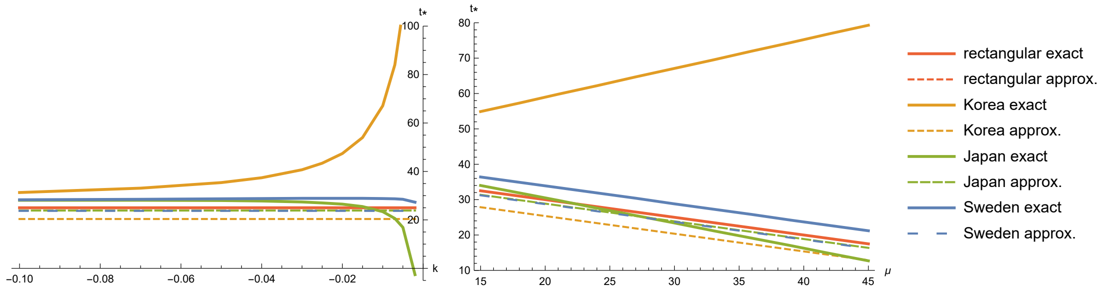

Figure 8: Time t^* of peak support ratio for concentrated mortality and three empirical life tables.

Figure 9 illustrates the dynamics of the support ratio over time for these three life tables and variations in k. For the computation of these support ratios we assume that working age lasts from $a_1 = 20$ to $a_2 = 65$. For the speed of fertility decline we use the values $k = -0.05, -0.02, -0.01$ and -0.005. Moreover, we fix the generation length at $\mu = 30$ years. As expected, and as we already saw previously, the curves become steeper for faster fertility declines, i.e. for higher absolute values of k. It seems counter intuitive that in the case of a slower fertility decline the support ratio $S(t)$ intersects the benchmark S_0 earlier. This is an artefact that arises because we standardize the time scale such that the net reproduction rate is one at $t = 0$. If two populations start at the same net reproduction rate above one but their fertility declines with different rates k, then they are actually situated at different times $t < 0$ in our formal model. Of course the time span until arriving at the first intersection of $S(t)$ with S_0 is shorter if the fertility decline happens at a higher rate.

From (8) we conclude that the sign of the difference $A_0 - A_{W,0}$ affects the sign of the derivative ${dt^*}/{dk}$ and, consequently, determines the direction of the shift of the peak for variations in k. The lower left panel shows the results obtained with the life table from Sweden in the period 2015–2020. With this life table the mean ages of the stationary total population A_0 and stationary working age population $A_{W,0}$ are almost equal. As a result, the peak does not shift appreciably with variations in k. The upper left panel illustrates results based on the life table of Japan in the period 2015–2020. In this case the difference $A_0 - A_{W,0}$ is positive and, consequently, the peak moves to the right, i.e. it is achieved later, as the speed of fertility increases. The opposite effect can be observed in the upper right corner illustrating the support ratio trajectories resulting from the life table of the Republic of Korea in the time period 1980–1985. The difference $A_0 - A_{W,0}$ is negative and the peak moves to the left as the speed of fertility decline accelerates. These findings regarding the sensitivity of the timing of the peak t^* with respect to variations in k are perfectly in line with what we would expect from (8).

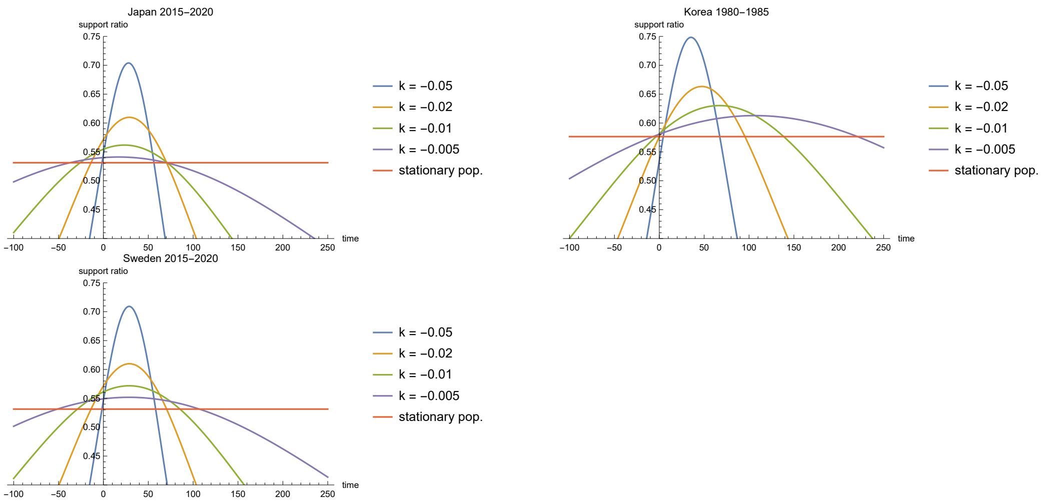

Figure 9: Support ratio over time for three different life tables and four different speeds of fertility decline compared to the constant support ratio of a stationary population.

For a detailed discussion of the influence of the speed of fertility decline k let us again have a look at figure 10. The dashed lines represent results obtained from real life tables. We used ochre for Korea, green for Japan and blue for Sweden. The curves for Japan and Sweden are similar to those obtained with the concentrated mortality schedule. In particular the proportional relationship between the total amount A and $\sqrt{-k}$ is remarkable. The results based on the Korean life table from the period 1980-1985 differ significantly from this. As we can see from figure 7, lifespan inequality is much larger for this life table and the absolute value of $A_0 - A_{W,0}$ in table 2 is also much larger than for the other two life tables. This causes pronounced deviations from the results obtained with the other survival schedules if the speed of fertility decline is low. For a more rapid decline of fertility, the results for Korea are close to the other three lines and we can conclude that the relationships between A, h and l vs. k are still in good order.

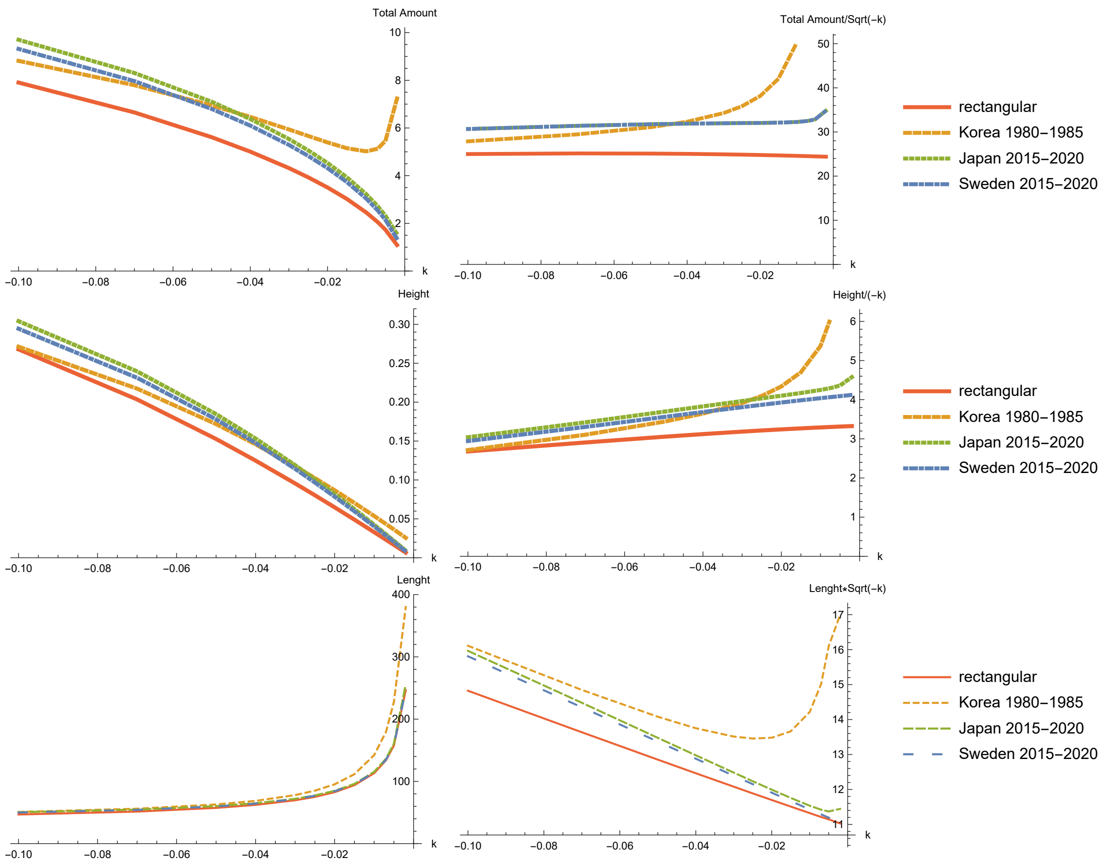

Figure 10: Total amount, height and length of the surplus in the support ratio for concentrated mortality and three empirical life tables and variations in k.

The support ratio trajectories depicted in figure 11 illustrate the influence of changes in the generation length μ. In this case, (9) gives us a hint as to how we expect variations in μ to affect t^*. We use life tables from the same three countries at the same time periods as before. The speed of fertility decline is constant at

$k = -0.01$ but generation length μ takes the values 20, 30 and 40. In reality the variation in the generation length is much smaller but we choose these values to make the sensitivity of the results visible. In all three cases, a higher generation length results in a flatter ascent and descent and a lower peak of the support ratio. In the lower left panel (Sweden 2015–2020, $A_0 - A_{W,0} \approx 0$) the peak moves to the left with an increase in the generation length. This is what we expect, since (9) shows that ${dt^*}/{d\mu}$ is negative if $A_0 - A_{W,0} = 0$. In the upper left panel (Japan 2015–2020, $A_0 - A_{W,0} > 0$) the peak moves to the left as the generation length increases. In the upper right panel (Republic of Korea 1980–1985, $A_0 - A_{W,0} < 0$) the peak moves to the right with an increase in generation length μ. These two graphs are again perfectly in line with our expectations resulting from (9).

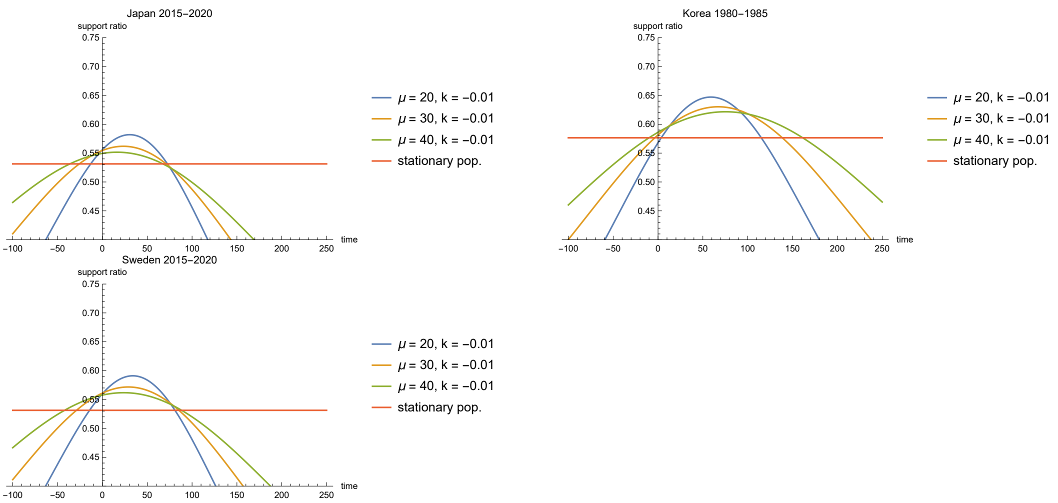

Figure 11: Support ratio over time for three different life tables and three different generation lengths compared to the constant support ratio of a stationary population.

For a more accurate assessment of the influence of μ on the dynamics of the support ratio we revisit figure 12. Again, the dashed lines result from real life tables and the mapping between colours and countries remains as before – ochre for Korea, green for Japan and blue for Sweden. As expected, the curves based on Japanese and Swedish life tables are again nearby those curves that stem from concentrated mortality and the proportional relationships between h vs. ${1}/{\mu}$ and l vs. $\sqrt{\mu}$ remain. Again the results we obtained with the Korean life table deviate and the deviation increases with increases in μ.

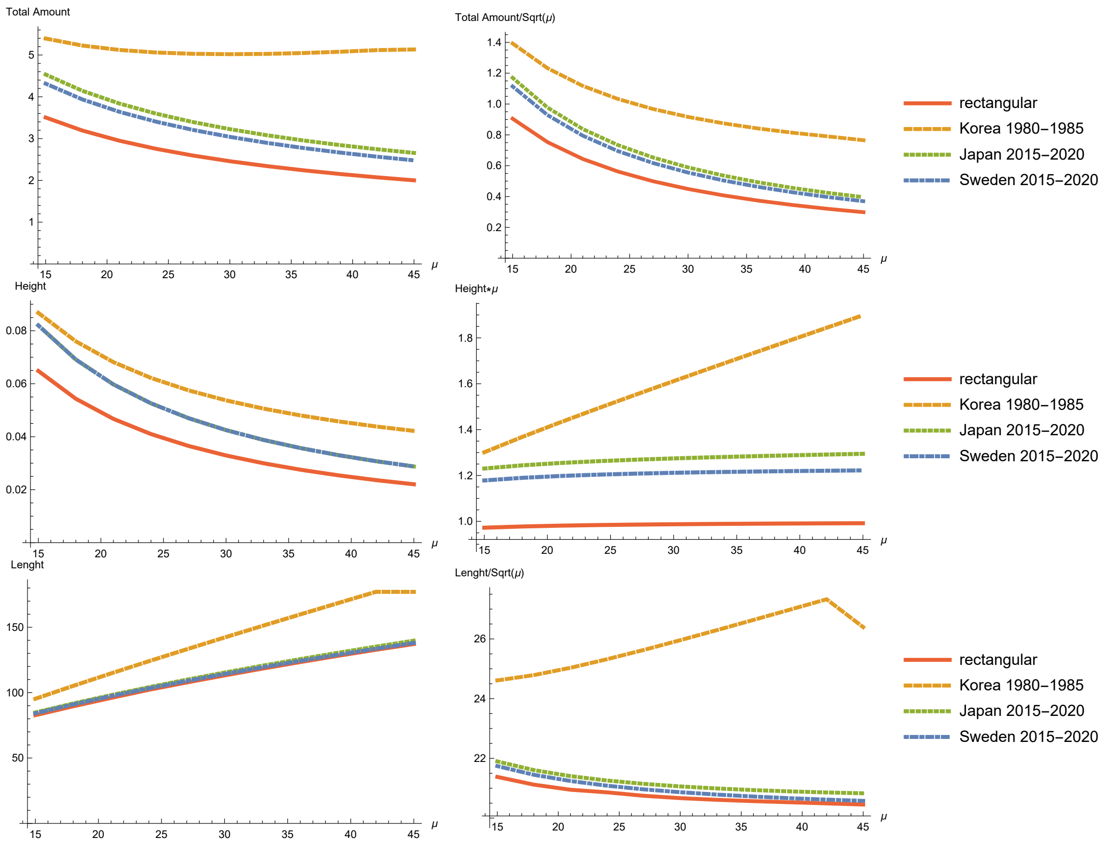

Figure 12: Total amount, height and length of the surplus in the support ratio for concentrated mortality and three empirical life tables and variations in μ.

6 Summary and Conclusions

The aim of our study is to analyse the temporary surplus in the share of the working age population that occurs when fertility declines in a population with initially high fertility. For this purpose we developed a mathematical model that resembles those characteristics of such a transitional population that are relevant for our research questions, but at the same time allows for meaningful and interpretable analytical solutions. The interplay of mathematical analysis with numerical evaluations gives insight into the dynamics of the support ratio when fertility changes at a constant rate over time. In particular, our analysis provides an in–depth understanding of the sensitivity of the results to variations in the numerical parameters as well as variations in the underlying survival functions.

We consider an age structure to be economically advantageous if the support ratio at a given time exceeds the support ratio of a stationary population based on the same survival function, i.e. $S(t) > S_0$. In the case of a population with an initially high but declining fertility, the favorable period begins at t_1 when $S(t)$ intersects S_0 from below and ends at t_2 when $S(t)$ intersects S_0 from above. The timing of the beginning t_1 depends strongly on the speed of fertility decline k but also on the generation length μ and the survival function $l(a)$. This may be before or after the time when the net reproduction rate NRR reaches replacement level.

The length l of this advantageous period is given by the difference $t_2 - t_1$ (13) and again depends on k, μ and $l(a)$. A faster decline in fertility as well as a shorter generation length shorten this time interval. The length is approximately inversely proportional to $\sqrt{-k}$ (24) and approximately proportional to $\sqrt{\mu}$.

We denote the difference between the support ratio peak $S(t^*)$ and S_0 as the height h of the surplus under consideration. This height also depends on k, μ and $l(a)$. A faster decline and a shorter generation length result in a higher peak. The height is approximately proportional to $-k$ (24) and approximately inversely proportional to μ.

We consider the area A between the support ratio of the pseudo–stable population $S(t)$ and the stationary population S_0 from time t_1 to time t_2 as a measure for the total amount of the gain from the surplus in the support ratio. This area is proportional to the height and length of the part of the trajectory $S(t)$ above S_0 (14). Consequently, the size (area) A is approximately proportional to $\sqrt{-k}$ and approximately indirectly proportional to $\sqrt{\mu}$ (23) and (24). The survival function $l(a)$ and the lower and upper bound of the working age also have an influence on the height h (11) and length l (13) and, consequently, on the size.

The initial net reproduction rate NRR in combination with the speed of fertility decline k has the strongest influence on the time it takes to arrive at the peak time t^*. This most important influence is hidden by choosing the time scale such that $t = 0$ when $NRR = 1$. Within this particular time scale, the timing of the peak t^*

is determined by the mean ages A_0 and $A_{W,0}$, as well as μ, k, t_0, $t_{W,0}$, σ_0^2 and $\sigma_{W,0}^2$ (prop. 1 and equ. (6)). If $A_0 = A_{W,0}$ the timing of the peak does not depend on k but only on A_0 and μ, if $A_0 \neq A_{W,0}$ changes in μ and k alter the timing of the peak (see figs. 9 and 11).

The benefit derived from an advantageous age structure lasts as long as the support ratio $S(t)$ is above the support ratio S_0 of the corresponding stationary population. Again, the initial net reproduction rate NRR in combination with the speed of fertility decline k has the strongest influence on the time it takes to arrive at the onset t_1 and end t_2 of the benefit. In addition, t_1 and t_2 are determined by the survival schedule $l(a)$, generation length μ, speed of fertility decline k, and the mean ages A_0 and $A_{W,0}$ (prop. 3 and equ. (13)). In the special case of concentrated mortality $t_{1,2}$ depend on μ, ω and k (21).

A Proofs

Derivation of equation (1)

If the number of births at time t is

$$B(t) = B(0) \exp\left(\frac{k}{2}t + \frac{k}{2\mu}t^2\right),$$

the number of births at time $t - a$ is

$$\begin{aligned} B(t-a) &= B(0) \exp\left(\frac{k}{2}(t-a) + \frac{k}{2\mu}(t-a)^2\right) \\ &= B(0) \exp\left(\frac{k}{2}t + \frac{k}{2\mu}t^2\right) \exp\left(-\frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{k}{\mu}at\right) \\ &= B(t)g(a,t) \end{aligned}$$

Proof of proposition 1

We use the moments (10) to express the mean ages of the stationary population and of the stationary working age population

$$A_0 = \frac{L_1}{L_0} \quad \text{and} \quad A_{W,0} = \frac{L_1^W}{L_0^W}\,. \tag{26}$$

The mean age of the pseudo–stable population is

$$
\begin{aligned}
A(t) &= \frac{\int_0^\omega aN(a,t)da}{\int_0^\omega N(a,t)da} = \frac{\int_0^\omega aB(t-a)l(a)da}{\int_0^\omega B(t-a)l(a)da} \\
&= \frac{\int_0^\omega aB(t)g(a,t)l(a)da}{\int_0^\omega B(t)g(a,t)l(a)da} = \frac{\int_0^\omega ag(a,t)l(a)da}{\int_0^\omega g(a,t)l(a)da}.
\end{aligned} \tag{27}
$$

Then we introduce the linear approximation for the function $g(a,t)$ in (1) around $a = 0$ (see Feichtinger and Vogelsang, 1978, equ. (6.3) and p. 41)

$$
g(a,t) = 1 - \frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a + o(k) \tag{28}
$$

and use the birth rate (2) to express the mean age of the pseudo–stable population as

$$
\begin{aligned}
A(t) = b(t)\int_0^\omega ag(a,t)l(a)da &\approx \frac{\int_0^\omega a\left(1 - \frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a\right)l(a)da}{\int_0^\omega \left(1 - \frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a\right)l(a)da} \\
&= \frac{L_1 - L_2\frac{k}{2} + L_3\frac{k}{2\mu} - L_2\frac{k}{\mu}t}{L_0 - L_1\frac{k}{2} + L_2\frac{k}{2\mu} - L_1\frac{k}{\mu}t}.
\end{aligned} \tag{29}
$$

Then we derive the time t_0 when the mean age of the pseudo–stable population equals the mean age of the stationary population. To do this, we use the mean age of the stationary population (26) and of the pseudo–stable population (29) and solve the equation $A_0 = A(t)$ for t,

$$
\begin{aligned}
\frac{L_1}{L_0} &= \frac{L_1 - L_2\frac{k}{2} + L_3\frac{k}{2\mu} - L_2\frac{k}{\mu}t}{L_0 - L_1\frac{k}{2} + L_2\frac{k}{2\mu} - L_1\frac{k}{\mu}t} \\
L_1L_0 + L_1^2\frac{k}{2} + L_1L_2\frac{k}{2\mu} - L_1^2\frac{k}{\mu}t &= L_1L_0 + L_0L_2\frac{k}{2} + L_0L_3\frac{k}{2\mu} - L_0L_2\frac{k}{\mu}t \\
(L_0L_2 - L_1^2)\frac{t}{\mu} &= \frac{L_1^2 - L_0L_2}{2} + \frac{L_0L_3 - L_1L_2}{2\mu}
\end{aligned}
$$

and obtain

$$
t_0 = -\frac{\mu}{2} + \frac{L_0L_3 - L_1L_2}{2(L_0L_2 - L_1^2)} = -\frac{\mu}{2} + A_0\theta \tag{30}
$$

with

$$
\theta = \frac{\frac{L_3}{L_1} - \frac{L_2}{L_0}}{2\sigma_0^2}. \tag{31}
$$

Analogously, we derive for the working age population

$$t_{W,0} = -\frac{\mu}{2} + A_{W,0}\theta_W\,. \tag{32}$$

This solution does not depend on k due to the linear approximation of $g(a,t)$. Thus, in a pseudo–stable population the time interval between the point in time when $NRR = 1$, which we normalize as $t = 0$, and the point in time t_0 when the mean ages A_0 and $A(t)$ are equal depends only on the generation length μ and on the mortality schedule $l(a)$ but not on k determining the speed of fertility decline. An increase in μ causes a decrease in t_0, i.e. the event $A(t) = A_0$ occurs earlier.

We introduce linear approximations for $A(t)$ and $A_W(t)$ at t_0 and $t_{W,0}$ applying (Feichtinger and Vogelsang, 1978, equ. (6.8)) for the rate of change of the mean age,

$$A(t) \approx A_0 + (t - t_0)\left(-\frac{k}{\mu}\right)\sigma_0^2 \tag{33}$$

$$A_W(t) \approx A_{W,0} + (t - t_{W,0})\left(-\frac{k}{\mu}\right)\sigma_{W,0}^2\,. \tag{34}$$

Now we use (33) and (34) to solve the equation $A(t) = A_W(t)$ for t to obtain

$$t^* = \frac{(A_0 - A_{W,0})\frac{\mu}{k} + t_0\sigma_0^2 - t_{W,0}\sigma_{W,0}^2}{\sigma_0^2 - \sigma_{W,0}^2}\,.$$

To obtain approximation (7) we insert (30) and (32) into the above expression for t^* to obtain

$$t^* = \frac{(A_0 - A_{W,0})\frac{\mu}{k} + A_0\theta\sigma_0^2 - A_{W,0}\theta_W\sigma_{W,0}^2}{\sigma_0^2 - \sigma_{W,0}^2} - \frac{\mu}{2}\,.$$

Then we use the assumption $A_0 = A_{W,0}$ and the approximations $\theta \approx \theta_W \approx 1$ to get

$$t^* = A_0 - \frac{\mu}{2}\,.$$

The relationship $\theta = \theta_W = 1$ is exact if mortality is concentrated at one fixed age and Fent (2023) shows that it is reasonable to assume $\theta \approx \theta_W \approx 1$ if lifespan inequality is moderate.

Proof of proposition 2

We introduce the linear quadratic approximation of (1) around $a = 0$,

$$g(a,t) = 1 - \frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a + \frac{1}{2}\left(-\frac{k}{2}a + \frac{k}{2\mu}a^2 - \frac{kt}{\mu}a\right)^2 + o(k^2)\,. \tag{35}$$

Then we substitute (35) into the equation for the support ratio (3),

$$\begin{aligned} S(t) = \Bigg[& L_0^W + \left(-\frac{k}{2} - \frac{kt}{\mu}\right) L_1^W + \left(\frac{k^2}{8} + \frac{k^2t^2}{2\mu^2} + \frac{k}{2\mu} + \frac{k^2t}{2\mu}\right) L_2^W \\ & + \left(-\frac{k^2t}{2\mu^2} - \frac{k^2}{4\mu}\right) L_3^W + \frac{k^2}{8\mu^2} L_4^W \Bigg] / \\ & \Bigg[L_0 + \left(-\frac{k}{2} - \frac{kt}{\mu}\right) L_1 + \left(\frac{k^2}{8} + \frac{k^2t^2}{2\mu^2} + \frac{k}{2\mu} + \frac{k^2t}{2\mu}\right) L_2 \\ & + \left(-\frac{k^2t}{2\mu^2} - \frac{k^2}{4\mu}\right) L_3 + \frac{k^2}{8\mu^2} L_4 \Bigg] , \end{aligned} \tag{36}$$

and evaluate this expression at t^* to obtain the maximum surplus $S(t^*) - S_0$ given in (11).

Proof of proposition 3

To find the points t_1 and t_2 when the support ratio of the pseudo–stable population, $S(t)$, is equal to the support ratio of the stationary population, S_0, we set $S(t)$ from (36) equal to $S_0 = L_0^W / L_0$ and solve for t and get

$$\begin{aligned} t_{1,2} = {} & 2\mu L_0 L_0^W (A_0 - A_{W,0}) \pm \Bigg\{ (L_0^W L_3 - L_0 L_3^W)^2 k^2 \\ & - \left[L_0^{W\,2}(L_2^2 - L_1 L_3) + L_0 L_0^W(-2L_2 L_2^W + L_1^W L_3 + L_1 L_3^W) + L_0^2 (L_2^{W\,2} - L_1^W L_3 W) \right] 4\mu k \\ & + \left[-L_0 L_3^W + L_0 L2^W \mu + L_0^W (L_3 - L_2 \mu) \right] k \Bigg\}^{1/2} / \left[(L_0^W L_2 - L_0 L_2^W) 2k \right] . \end{aligned} \tag{37}$$

Proof of proposition 4

With (15) we get the ith moments

$$L_0 = \omega, L_1 = {}^{\omega^2}\!/_2, L_2 = {}^{\omega^3}\!/_3, L_3 = {}^{\omega^4}\!/_4,$$

and life expectancy, mean age and variance of the stationary population and θ

$$e_0^0 = \omega, A_0 = {}^{\omega}\!/_2, \sigma_0^2 = {}^{\omega^2}\!/_{12} \text{ and } \theta = 1.$$

If we define the working age interval $W = [\underline{w}, \overline{w}]$, the i-th moments for the working age population become

$$L_i^W = {}^{(\overline{w}^{i+1} - \underline{w}^{i+1})}\!/_{i+1},$$

which results in

$$A_{W,0} = {}^{(\underline{w}+\overline{w})}\!/_2, \; \sigma_{W,0}^2 = {}^{(\overline{w}-\underline{w})^2}\!/_{12}, \text{ and } \theta_W = 1.$$

From this and equations (30) and (32) we get t_0 ($t_{W,0}$), the time when the mean age of the pseudo–stable (working age) population equals the mean age of the stationary population,

$$t_0 == \frac{\omega - \mu}{2} \text{ and} \tag{38}$$

$$t_{W,0} == \frac{\underline{w} + \overline{w} - \mu}{2}. \tag{39}$$

Then we insert (38) and (39) into (6) and obtain (16). Assuming that the working age interval W is symmetric around $^{\omega}/_{2}$ means $\overline{w} = \omega - \underline{w}$. Inserting this into (16) leads to (17). We sum up, that under these simplifying assumptions we get $t_0 = t_{W,0} = t^* = {}^{(\omega-\mu)}/_{2}$ which only depends on μ and ω but not on k.

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