The Average Uneven Mortality index: Building on the $e^\dagger$ measure of lifespan inequality

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Abstract

BACKGROUND
In recent years, lifespan inequality has become an important indicator of population health. Uncovering the statistical properties of lifespan inequality measures can provide novel insights on the study of mortality.

METHODS
We revisit the $e^\dagger$ measure of lifespan inequality, introduced in Vaupel and Canudas-Romo (2003). Leveraging a result first noted in Schmertmann (2020), we derive an upper bound for $e^\dagger$. This finding motivates us to introduce the "Average Uneven Mortality" (AUM) index, a normalized version of the $e^\dagger$ measure that can be meaningfully compared across countries and over time.

RESULTS
The use of the AUM index is illustrated through an application to observed period and cohort death rates from the Human Mortality Database. We explore the behavior of the index across age and over time, and we study its relationship with life expectancy. The AUM index at birth declined over time until the 1950s, when it reverted its trend; also, the index generally increases with age.

CONTRIBUTION
We elaborate on Vaupel and Canudas-Romo's $e^\dagger$ measure, deriving its upper bound. We exploit this result to introduce a novel mortality indicator, which provides a new perspective on the historical evolution of lifespan inequality. We also develop novel routines to compute $e^\dagger$ and the standard deviation of lifetimes σ_T from death rates, which are possibly more precise than available software, particularly for calculations involving older ages.

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1 Relationships

An upper bound for $e^\dagger$

Let $e^\dagger$ denote the average number of life-years lost as a result of death (Vaupel and Canudas-Romo, 2003). Schmertmann (2020) showed that Vaupel and Canudas-Romo's $e^\dagger$ equals the covariance between the time to death T and its transformation through its own cumulative hazard function, i.e. $\Lambda(T)$:

$$e^\dagger = \text{Cov}[T, \Lambda(T)]\,.$$

Here, we highlight a new result. This covariance relationship implies an upper bound on $e^\dagger$. In particular, it holds that:

$$0 \leq e^\dagger \leq \sigma_T\,, \tag{1}$$

i.e., $e^\dagger$ is bounded from above by the standard deviation σ_T of the time to death T. This result holds more generally for the indices computed conditionally on surviving until any given age a:

$$0 \leq e^\dagger(a) \leq \sigma_T(a)\,. \tag{2}$$

The AUM, a novel mortality indicator

We leverage the finding of an upper bound on $e^\dagger$ to introduce a novel mortality indicator. We define the "Average Uneven Mortality" (AUM) index as the correlation coefficient between T and $\Lambda(T)$:

$$\text{AUM} = \text{Corr}[T, \Lambda(T)] = \frac{e^\dagger}{\sigma_T}\,. \tag{3}$$

The AUM is a novel mortality index that can be used to study lifespan inequality and mortality patterns. It is a normalized version of $e^\dagger$ that ranges between 0 and 1; as such, it provides a rigorous assessment of lifespan inequality comparisons across countries and over time (since it ranges on a fixed, rather than time-varying, support). Moreover, the index is equal to 1 if and only if T has an exponential distribution with parameter θ ($T \sim Exp(\theta)$). As a consequence, the index can help determine whether the hazard rate is constant (AUM $= 1$) or varies with age (AUM < 1). This can be particularly useful in the analysis of the tail of the survival distribution, conditional to survival to different ages.

Let $\dot{\text{AUM}}$ denote the partial derivative of AUM with respect to calendar year[1]. Then, the relative derivative of AUM is such that:

$$\begin{aligned} \dot{\log(\text{AUM})} > 0 \quad &\Longleftrightarrow \quad \dot{\log(e^\dagger)} > \dot{\log(\sigma_T)} \\ \dot{\log(\text{AUM})} < 0 \quad &\Longleftrightarrow \quad \dot{\log(e^\dagger)} < \dot{\log(\sigma_T)}\,, \end{aligned}$$

that is, the relative change of the AUM index is positive (negative) when the relative change in $e^\dagger$ is greater (lower) than the relative change in σ_T.

2 Proofs

Consider a population whose individuals have the same initial exact age a and face identical force of mortality $\lambda(t)$ in future years $t \geq 0$ beyond a. Let T thus be a non-negative random variable denoting time to death, which is distributed according to the probability density function $f(t)$. Let $S(t) = \exp[-\int_0^t \lambda(u)du] = \exp[-\Lambda(t)]$ denote the survival function, i.e. the fraction of the

[1] In the following, a dot over a function will denote its partial derivative with respect to calendar year y (which may refer to either a given time period or birth cohort), but we drop the notation y to ease readability.

population expected to survive at least t years. Note that this scenario can be rewritten in terms of conditional random variable and conditional density function of the original population, starting from age 0 rather than age a (see Supplementary Materials).

Let $e^\dagger = \int_0^\omega e(u)f(u)du$ denote the life lost to mortality (Vaupel and Canudas-Romo, 2003), where $e(u) = \int_u^\omega S(t)dt/S(u)$ denotes the remaining life expectancy at age u, and ω the highest age attained in the population.

We start from the relationship shown by Schmertmann (2020):

$$e^\dagger = \mathrm{Cov}[T, \Lambda(T)]\,, \tag{4}$$

i.e. the equality between $e^\dagger$ and the covariance between T and its transformation through its own cumulative hazard function (for an alternative proof of this relationship, see Supplementary Materials). Rewriting (4) in terms of correlation rather than covariance produces:

$$e^\dagger = \mathrm{Corr}[T, \Lambda(T)] \cdot \sigma_T \cdot \sigma_{\Lambda(T)}. \tag{5}$$

Now, we consider the well-known fact that, for any hazard function $\lambda(t)$, $\Lambda(T) \sim Exp(1)$ (for a short proof, see Supplementary Materials). This implies that $\sigma_{\Lambda(T)} = 1$, yielding the interesting new expression:

$$e^\dagger = \mathrm{Corr}[T, \Lambda(T)] \cdot \sigma_T. \tag{6}$$

Furthermore, since T and $\Lambda(T)$ are two non-negative random variable, their correlation is also non-negative, i.e. $\mathrm{Corr}[T, \Lambda(T)] \in [0, 1]$. As such:

$$0 \leq e^\dagger \leq \sigma_T\,. \tag{7}$$

Recalling that the analysis is conditional on survival to the assumed starting age a, we can generalize as:

$$0 \leq e^\dagger(a) \leq \sigma_T(a)\,, \tag{8}$$

where $e^\dagger(a)$ is the life lost to deaths after age a, and $\sigma_T(a)$ the standard deviation of time to death after age a, both conditional on survival to age a.

Next, we define the "Average Uneven Mortality" (AUM) index as the correlation coefficient between T and $\Lambda(T)$, i.e. $\mathrm{AUM} = \mathrm{Corr}[T, \Lambda(T)]$. Rewriting the correlation in terms of covariance, and recalling that $\sigma_{\Lambda(T)} = 1$, we can rewrite AUM as:

$$\mathrm{AUM} = \frac{\mathrm{Cov}[T, \Lambda(T)]}{\sigma_T \cdot \sigma_{\Lambda(T)}} = \frac{e^\dagger}{\sigma_T}\,. \tag{9}$$

Leveraging the upper and lower bounds of $e^\dagger$ in Eq. (7), it follows that:

$$0 < \mathrm{AUM} \leq 1\,, \tag{10}$$

with the result again holding more generally conditionally on survival to any given age a.

We now prove that AUM is equal to 1 if and only if T has an exponential distribution. On one hand, if $T \sim Exp(\theta)$, then $\Lambda(T) = T \cdot \theta$ and $\mathrm{Corr}[T, \Lambda(T)] = \mathrm{AUM} = 1$. On the other hand, if $\mathrm{AUM} = \mathrm{Corr}[T, \Lambda(T)] = 1$, then $\Lambda(T) = a + bT$. Since $\Lambda(0) = 0$, then $a = 0$, implying that $\Lambda(T) = bT$ and $S(T) = e^{-bt}$, i.e. $T \sim Exp(b)$.

Finally, let us derive the partial derivative of AUM with respect to calendar year y:

$$\dot{\mathrm{AUM}} = \frac{\dot{e^\dagger}\,\sigma_T - \dot{\sigma_T}\,e^\dagger}{\sigma_T^2}\,, \tag{11}$$

from which follows that the partial (and relative) derivative of AUM is positive (negative) when the numerator of Eq. (11) is positive (negative). Rearranging terms (and since both measures are non-negative):

$$\dot{\text{AUM}} > 0 \iff \log(\dot{\text{AUM}}) > 0 \iff \log(\dot{e^\dagger}) > \log(\dot{\sigma_T})$$
$$\dot{\text{AUM}} < 0 \iff \log(\dot{\text{AUM}}) < 0 \iff \log(\dot{e^\dagger}) < \log(\dot{\sigma_T})\,.$$

3 Related results

The closest link to the relationships that we present in this paper is provided by Schmertmann (2020) in the framework of revivorship models, where the author derives the equality between $e^\dagger$ and the covariance between T and its transformation through its own cumulative hazard function. The cumulative hazard function is not frequently used by demographers. One recent exception is provided by Ullrich et al. (2022): the authors introduce a new longevity measure based on the cumulative hazard. The proposed death expectancy H_1 indicator corresponds to the age at which the cumulative hazard is equal to one (or equivalently, the survival function is about 36.8%). The authors argue that the H_1 measure could be used as a dynamic threshold age for the oldest-old.

In demography, there exists a variety of indicators that are used to summarize the age-at-death distribution. Several indicators focus on the first moment, or location, of the distribution, i.e. the so-called "central longevity indicators" (mean, median, and modal ages at death, Canudas-Romo, 2010; Cheung et al., 2005). Another class of indicators is used to study the second moment, or scale, of the distribution. Both absolute and relative measures of the variability of the distribution are used for this purpose. Examples of absolute indices are the variance and the $e^\dagger$ measures, while the life-table entropy, the coefficient of variation, and the Gini coefficient are examples of relative measures. Typically, relative measures of variability are computed by dividing an absolute variability measure by life expectancy. Up to our knowledge, the AUM index is among the very firsts mortality indicators that go beyond these two classes of indicators, by analysing the ratio of two absolute measures of variation. One recent proposal in this direction is the ratio of expansion to compression measure, which considers the $e^\dagger$ components before and after the threshold age at death (Zhang and Li, 2020).

The life-table entropy measure can be partly related to the AUM index, since both measures share the same numerator. The life-table entropy is a relative measure of variability of the age-at-death distribution compared to life expectancy at birth (Demetrius, 1974; Keyfitz, 1977; Leser, 1955). However, the difference in the denominator of the two measures results into two different interpretation of the indices: the entropy measures the (relative) variability of the distribution, while the AUM measures how close the distribution is from having constant mortality. Both indices can nonetheless be used for the study of lifespan inequality, with the AUM providing an innovative perspective in terms of normalized lifespan inequality. Another recent attempt in this direction was made by Permanyer and Shi (2022), who introduced normalized lifespan inequality to explicitly consider that life expectancy has been increasing at a faster pace than maximal length of life. For any given year, the authors compute normalized lifespan inequality by dividing lifespan inequality indices by their maximum value under an hypothetical distribution with life expectancy equal to the observed one.

4 Applications

Here we illustrate the use of the AUM index with an application to observed period and cohort death rates (obtained by dividing deaths by exposures), as well as to period life-table death rates. All data were retrieved from the Human Mortality Database (2023, henceforth HMD). Routines for deriving these results were developed in R (R Core Team, 2022) are are available in the Supplementary Materials as well as in a [blind for peer review] open-access repository[2]. The analytical formulas that we derived and employed for the implementation of our routines are available in the Supplementary Materials. Unless otherwise specified, all mortality measures are computed from age zero.

We start by investigating the temporal evolution of the AUM index in observed period death rates for two populations: Swedish females and Italian males. Figure 1 shows the $e^\dagger$, σ_T and AUM measures over time for both populations. In the left panels, we can observe the well-known reduction of lifespan inequality over most of the period analysed, coinciding with the increase in life expectancy (see, e.g., Aburto et al., 2020; Edwards and Tuljapurkar, 2005; Wilmoth and Horiuchi, 1999). The two graphs show that $e^\dagger$ never exceeds σ_T, in agreement with our derived upper bound of $e^\dagger$. The right panels show that the AUM index declined rather consistently from 1751 until the 1950s, when it reached a minimum value. Thereafter, a reversal of the decreasing trend is observable. Sudden increases in the index are visible in correspondence with the Spanish flu (for both populations) and the two World Wars (for males only). From the 2000s, the index displays a rather constant behavior.

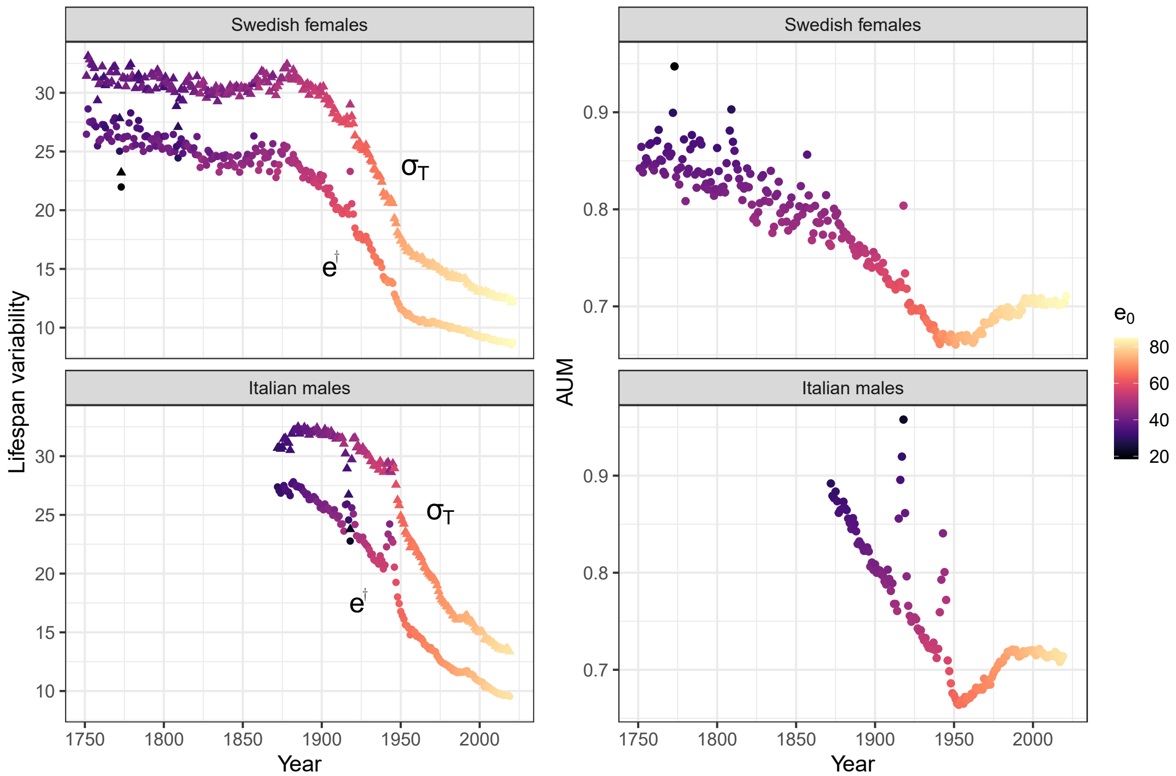

Figure 1: Evolution of the $e^\dagger$ (points) and σ_T (triangles) lifespan variability measures (left panels), and of the AUM index (right panels) over time for Swedish females and Italian males, 1751–2021. Colors correspond to different levels of life expectancy at birth (e_0).
Source (all figures and tables): Authors' own elaborations on data from the HMD (2023).

Analysing the temporal trend of the AUM index across the 41 populations available in the HMD by sex provides additional insights. Figure 2 shows a rather substantial overlap in the decrease of the AUM index across sexes until the 1950s (except during the two World Wars). From the 1950s onward, as the AUM index stopped declining and started its increase, a marked departure from the overlap between sexes occurred, with male populations generally characterized by greater

[2] Available at: `https://osf.io/fj94p/?view_only=65ef7ac73dbc46318be3284d55722214`

values of the AUM. It is further worth noticing that, for several female populations, the decline of the AUM halted and reversed somewhat later than the 1950s.

Figure 2: Evolution of the AUM index over time for 41 female (purple) and male (orange) populations, 1751–2021.

How can we interpret these trends of the AUM index? Formally, as we have shown, the AUM decreases (increases) when the relative change in $e^\dagger$ is lower (greater) than the relative change in σ_T. This means that throughout most of the period analysed (1751–2021), the relative change (generally, reduction) in σ_T was greater than the one in $e^\dagger$; however, a reversal of this trend occurred around the 1950-60s. This period is often identified with a transition to a new mortality regime, characterized by an acceleration of mortality improvements at older ages (Kannisto et al., 1994; Vaupel et al., 1998; Wilmoth and Horiuchi, 1999) and a more pronounced shifting dynamic of the age-at-death distribution (Bergeron-Boucher et al., 2015). From this perspective, the AUM index provides insights on the transition from mortality compression to mortality shifting (Janssen and de Beer, 2019).

An alternative interpretation of these findings can be made considering that the AUM is the normalized version of $e^\dagger$. The normalization implies that AUM takes values on a fixed support, i.e. between 0 and 1, for all years and populations. Conversely, $e^\dagger$ can vary on very different supports. Consider, for example, the top left panel of Figure 1. Until the 1900s, $e^\dagger$ could take values up to approx. 33 years; in 2000, its maximum value would have been approx. 13 years. The fact that $e^\dagger$ (and several other lifespan inequality measures) has a time-varying support may bias our assessment of lifespan inequality trends. Conversely, the fixed support of the AUM index allows for a more meaningful comparison of the index across countries and over time. Our findings suggest that the normalized years of life lost have not continued to decrease throughout the time period analysis (as suggested instead from the historical evolution of lifespan inequality measures, see e.g., Edwards and Tuljapurkar, 2005; Wilmoth and Horiuchi, 1999): as lifespan inequality continued to decrease, normalized lifespan inequality started to increase around the 1950-60s. This is further illustrated by Figure 3 which shows, for the 41 female populations of the HMD, the relationship between the AUM index and two indices of (absolute and relative) lifespan variability: the $e^\dagger$ and the entropy of the life table. The figure clearly shows that, as lifespan inequality continued to reduce over time, the AUM index declined for most of the period considered, reaching a minimum for values of $e^\dagger$ and the entropy around 13.5 and 0.2, respectively, when it then started to increase.

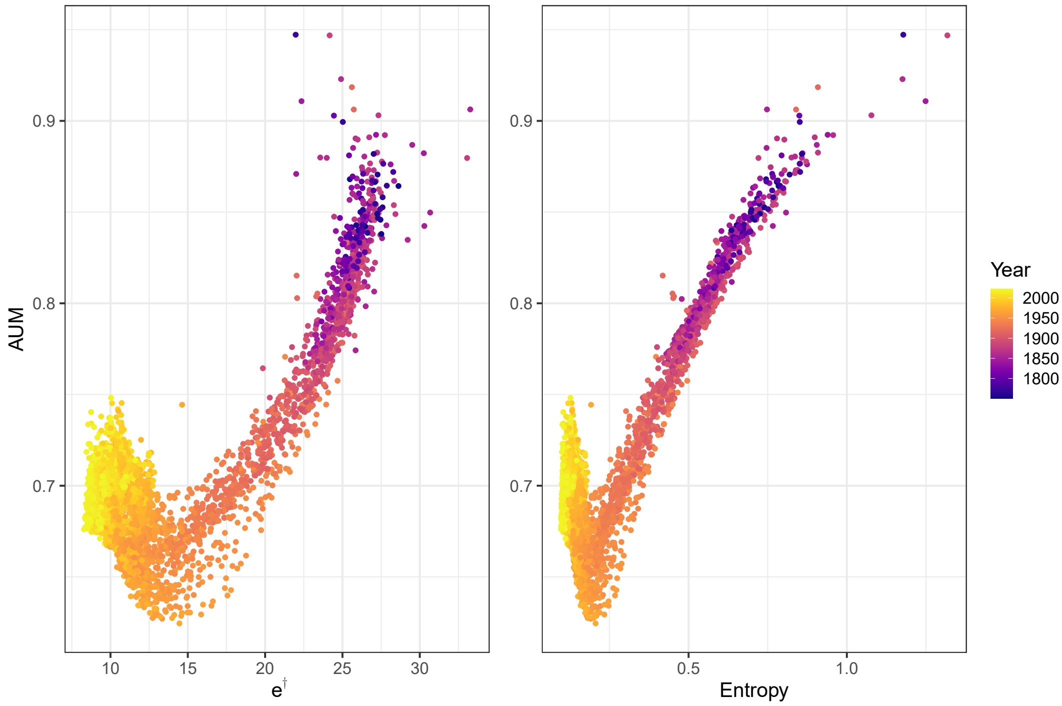

Figure 3: Relationship between the AUM index and absolute and relative lifespan variability, measured with the $e^\dagger$ (left) and the life table entropy (right), respectively, for 41 female populations from 1751 to 2021. Colors correspond to different calendar years.

The reduction of the AUM index throughout most of the analysed period can also be interpreted with respect to how close the distribution is to an exponential one (i.e. one with constant mortality). On the one hand, the density of the exponential distribution is monotonically decreasing, having its maximum at age zero. On the other hand, the age-at-death distribution is typically bimodal, with one mode in infancy and another in late life (Kannisto, 2001). It is probably not surprising that the highest values of the AUM index were observed in the past, when a significant number of deaths were occurring at infant and childhood ages: indeed, the corresponding distribution of deaths was characterized by high and monotonically decreasing values at the youngest ages, somewhat closer to the shape (at least in those early ages) of the exponential distribution. Mortality improvements at these ages throughout subsequent decades decreased the relative importance of deaths at these ages, with more and more deaths occurring at older ages. These improvements in infant mortality are reflected in the reduction of the AUM.

Next, we analyse the relationship between the AUM index and period life expectancy at birth. Figure 4 shows this relationship for the 41 populations in the HMD, by sex. For low levels of life expectancy, there is a linear negative relationship between the two measures. However, there appears to exist a threshold level of life expectancy (around age 70 and 60 for females and males, respectively) at which the negative relationship ceases to hold. For females, a positive relationship emerges above the threshold, whereas for males, the relationship appears to be more erratic. It should be noted that, due to data limitations regarding historical data, few data points are available for older periods, which are characterized by lower levels of life expectancy; as such, the strong linear relationship observed for low levels of life expectancy could be partially due to data limitations.

We now move to the same analysis using cohort instead of period observed death rates. Figure 5

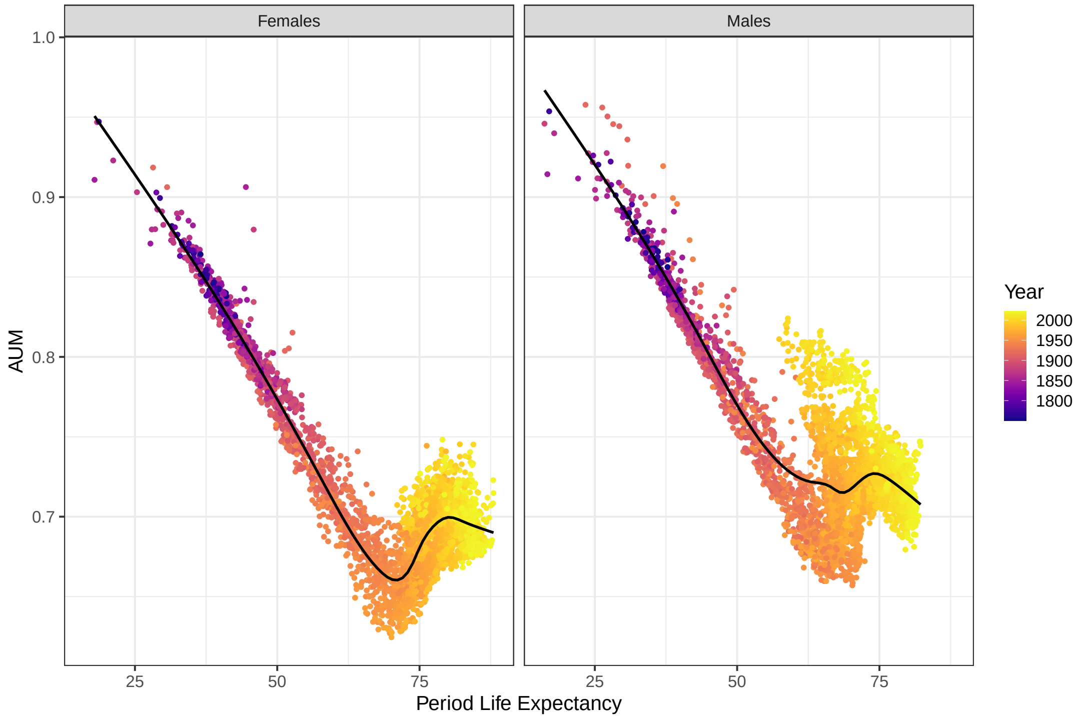

Figure 4: Relationship between the AUM index and period life expectancy at birth for 41 female (left) and male (right) populations from 1751 to 2021. Colors correspond to different calendar years. The black line corresponds to smoothing the observed data using a cubic regression spline (using the `gam` function of the `mgcv` package (Wood, 2017))

shows the relationship between the AUM index and cohort life expectancy at birth for six female and male populations with a long mortality data series[3]. Unlike the period analysis, one observes a lack of reversal of the linear relationship between the two measures. For cohorts, the AUM tends to linearly decrease over increasing values of cohort life expectancy (typically belonging to the more recent cohorts).

Finally, we turn to study the behavior of the AUM index at all ages, focusing on a single population only, namely Swedish females. In addition to computing the AUM using observed death rates, we also employ period life-table death rates. The reason for doing so is that observed and life-table death rates differ at the oldest ages, since the latter are smoothed at older ages using the Kannisto model of mortality (see Wilmoth et al., 2021, p. 34).

Figure 6 shows the results of the age analysis. In general, we observe that the AUM index tends to increase with age (declining in the first few years of life in the oldest periods considered). Importantly, the figure highlights the difference of the AUM index at the oldest ages. When the index is computed on observed rates (left panel), there is high variability of the AUM estimate at the oldest ages, reflecting the variability of the underlying rates; conversely, when life-table death rates are used as input, the AUM index approaches the value of 1. This was an expected result, since life-table death rates are smoothed at older ages according to a logistic pattern (the Kannisto model), which is characterized by a flatter and flatter hazard function that mimics an exponential (constant) hazard (whose AUM value is exactly 1).

It is noteworthy to further mention here that conventional routines generally employed to calculate life table variability measures (such as those available in the `LifeIneq R` package, Riffe et al., 2023) return AUM estimates that exceed the upper bound of 1 at the very older ages, suggesting

[3] The six populations are those of France, Italy, Finland, Denmark, Sweden, and the Netherlands.

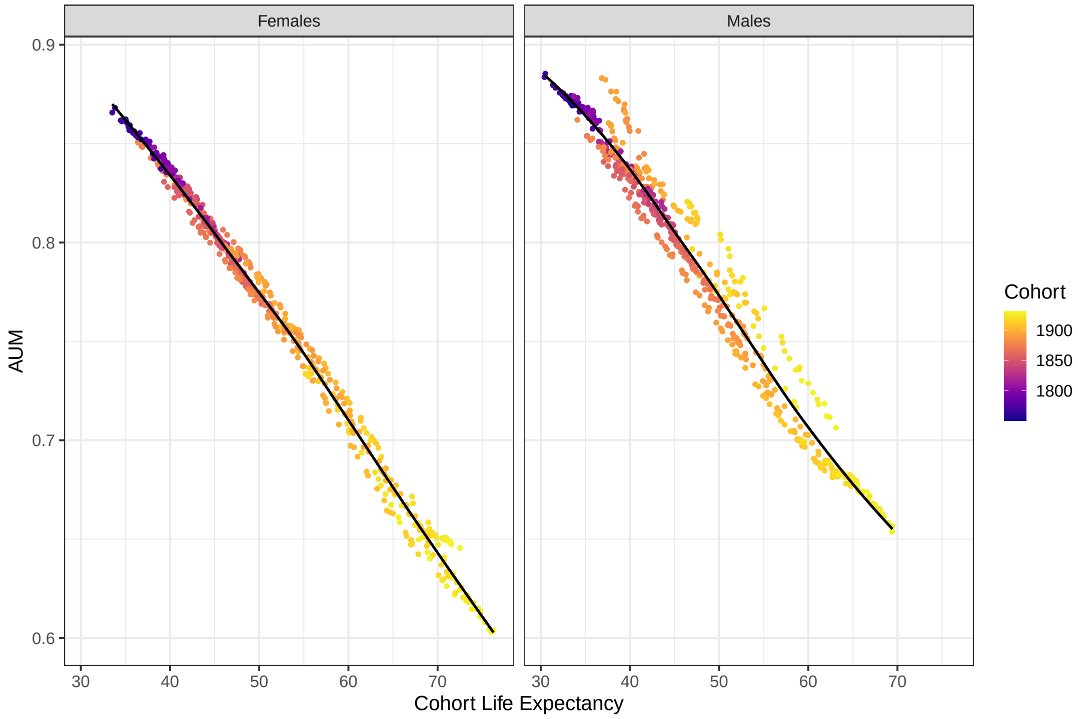

Figure 5: Relationship between the AUM index and cohort life expectancy at birth for 6 female (left) and male (right) populations from 1751 to 1930. Colors correspond to different birth cohorts. The black line corresponds to smoothing the observed data using a cubic regression spline (using the `gam` function of the `mgcv` package (Wood, 2017)).

some potential bias in the estimation of $e^\dagger$ or σ_T, or both, at older ages. Please refer to the Supplementary Materials for a comparison analysis between our AUM estimates and those derived from conventional routines.

5 Discussion

In this paper, we have elaborated on the $e^\dagger$ measure of lifespan inequality introduced by Vaupel and Canudas-Romo (2003). Leveraging a recent result noted in Schmertmann (2020), we derived the upper bound of $e^\dagger$. This is, up to our knowledge, a novel and important result. It is intriguing that the upper bound of $e^\dagger$ is another absolute measure of lifespan inequality – the standard deviation of ages at death in the population (σ_T). Even more intriguing, we have shown that $e^\dagger$ reaches its upper bound if and only if the underlying age-at-death distribution is exponential.

The upper bound of $e^\dagger$ stimulated us to introduce the "Average Uneven Mortality" (AUM) index, a new mortality index that can be used to study mortality across age and over time. The index has two closely related interpretations. On the one hand, it measures the linearity of the relationship between the random variable T and its cumulative hazard function; on the other hand, and equivalently, it measures the distance of the age-at-death distribution from an exponential one, or the distance of the hazard function from a constant ("even") hazard. The AUM is a relative index, bounded between 0 and 1, and it is obtained by dividing two absolute measures of variation of the age-at-death distribution. This is one among the very first proposals to build a mortality indicator based on (the ratio of) two lifespan inequality measures. Other relative indicators of mortality employed in the literature are computed by dividing an absolute variability measure by life expectancy (e.g. life-table entropy, coefficient of variation, Gini coefficient).

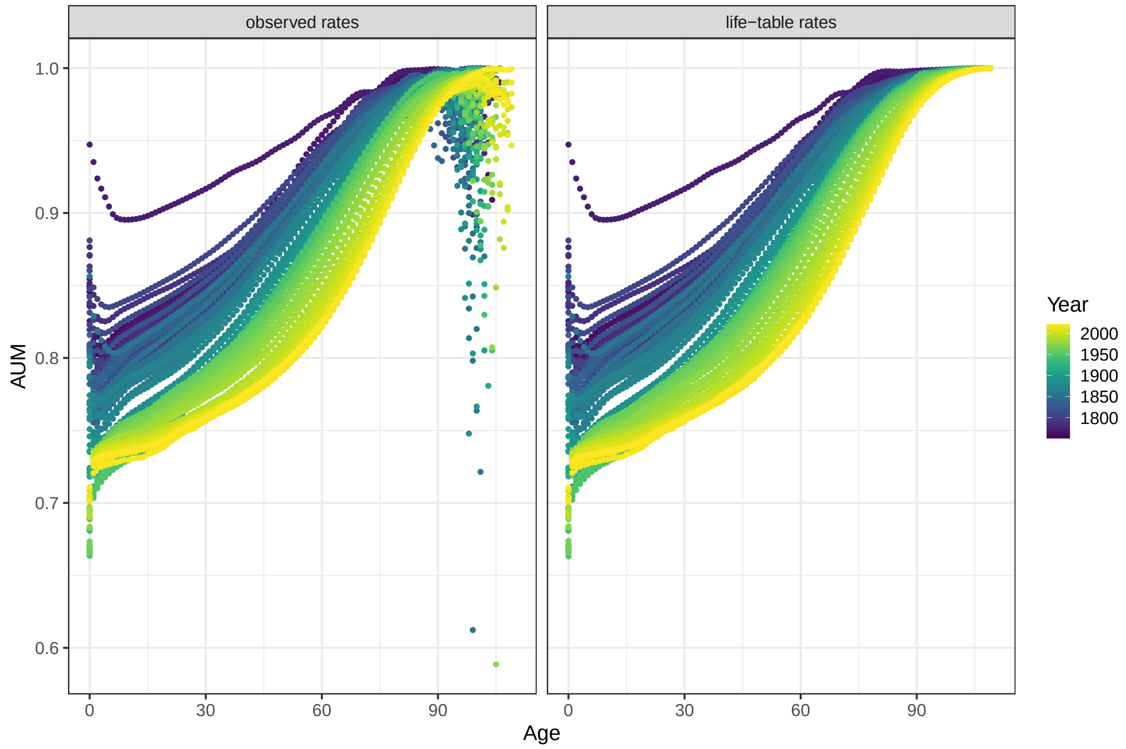

Figure 6: AUM index over ages 0-110+ for Swedish females for the years 1751–2021 computed on observed rates (left panel) and life-table rates (right panel).

Introducing a novel mortality index raises the natural "so what?" question. We believe that the AUM index provides novel insights on the study of human mortality age patterns and time developments. Importantly, the AUM index takes values on a fixed support, suggesting that comparison across countries and over time are more meaningful than when one uses an indicator whose range can change over time (such as $e^\dagger$). Indeed, in applied statistics, the correlation coefficient is generally favoured to the covariance, since the latter tells little about the strength of the dependence between two random variables. The analysis of the normalized $e^\dagger$ suggests that the decrease of lifespan inequality has reversed its secular decline in most recent decades. Our findings align well with those of Permanyer and Shi (2022), who also observed that declines of normalized lifespan inequality stopped and even reversed at high levels of life expectancy. Furthermore, while it is well-known that lifespan inequality measures are highly correlated between each other (see, e.g. Van Raalte and Caswell, 2013; Wilmoth and Horiuchi, 1999), the evolution of the AUM index shows that the relative change of $e^\dagger$ and σ_T differed over time: the relative change (typically reduction) in σ_T was greater than the one of $e^\dagger$ for most of the period that we analysed. The 1950s marked a clear reversal of this trend, likely connected to accelerating mortality improvements at older ages and related shifting of the age-at-death distribution.

The relationship between the AUM index and life expectancy at birth provides an interesting perspective on the evolution of the age-at-death distribution. We found a linear negative relationship between the AUM and life expectancy, which ceased to hold only at high levels of period life expectancy. Interestingly, this disruption did not occur in the analysis of cohort mortality data. Due to the limitation of such data, we cannot know whether this disruption will materialise for more recent birth cohorts, or if it is a specific feature of the most recent period death rates. If the second hypothesis were to hold true, it would imply that information on the scale of the age-at-death distribution would be predictive of its location, with important consequences for mortality analysis and forecasting.

The AUM index can also be employed to study the mortality age-pattern. In our analysis, we observed that the AUM index generally increases over age. In the analysis of period life tables (with modelled death rates), the index always approaches its upper limit at older ages, signaling that the hazard function resembles a constant exponential hazard. When observed death rates are employed instead, there is significant variability in the estimate of the AUM at older ages. Clearly, there is great potential in employing the AUM index for detecting mortality deceleration, and eventually the existence of a mortality plateau; our proposed indicator could thus contribute to the current debate about the mortality plateau at the oldest ages (see, e.g. Barbi et al., 2018; Dang et al., 2023; Gampe, 2021; Newman, 2018). The quantification of the statistical uncertainty associated with the estimated AUM estimate should be a critical aspect to inform such analysis, especially for small sample sizes, and we plan to pursue this in our future research.

An important contribution of our work relates to the software routines that we have developed to calculate the AUM index. Initially, we started by computing the AUM using standard and available routines for calculating lifespan inequality measures from a life table. The results that we obtained were unexpected, as the AUM was exceeding its upper bounds at older ages – an empirical result that contradicted our theoretical findings. As such, we decided to implement new routines, based on the formulas that we derived in this paper. The new empirical estimates that we obtained did not present such anomalies. While we cannot be certain, evidence presented in the Supplementary Materials suggests that our routines for computing $e^\dagger$ and σ_T improve estimation precision with respect to conventional routines, particularly for the older ages. Our routines are publicly available in the Supplementary Materials accompanying this article as well as in a public repository, and we hope that further computational efforts will be directed to assess estimation accuracy of lifespan variability measures at the oldest ages.

Finally, analysing the AUM index can be related to the more general study of the shape of the age-at-death distribution, which has gained increasing attention in most recent decades (for a recent review, see, e.g., Bonetti et al., 2021). We believe that this novel mortality indicator can provide additional insights on human mortality, enlarging the toolbox of available methods for the analysis of mortality developments.

References

Aburto, J. M., Villavicencio, F., Basellini, U., Kjærgaard, S., and Vaupel, J. W. (2020). Dynamics of life expectancy and life span equality. *Proceedings of the National Academy of Sciences*, 117(10):5250–5259.

Barbi, E., Lagona, F., Marsili, M., Vaupel, J. W., and Wachter, K. W. (2018). The plateau of human mortality: Demography of longevity pioneers. *Science*, 360(6396):1459–1461.

Bergeron-Boucher, M.-P., Ebeling, M., and Canudas-Romo, V. (2015). Decomposing changes in life expectancy: Compression versus shifting mortality. *Demographic Research*, 33:391–424.

Bonetti, M., Gigliarano, C., and Basellini, U. (2021). The Gini Concentration Index for the Study of Survival. In *Gini Inequality Index*, pages 107–124. Chapman and Hall/CRC.

Canudas-Romo, V. (2010). Three measures of longevity: Time trends and record values. *Demography*, 47(2):299–312.

Cheung, S. L. K., Robine, J.-M., Tu, E. J.-C., and Caselli, G. (2005). Three dimensions of the survival curve: Horizontalization, verticalization, and longevity extension. *Demography*, 42(2):243–258.

Dang, L. H. K., Camarda, C. G., Meslé, F., Ouellette, N., Robine, J.-M., and Vallin, J. (2023). The question of the human mortality plateau: Contrasting insights by longevity pioneers. *Demographic Research*, 48:321–338.

Demetrius, L. (1974). Demographic parameters and natural selection. *Proceedings of the National Academy of Sciences*, 71(12):4645–4647.

Edwards, R. D. and Tuljapurkar, S. (2005). Inequality in life spans and a new perspective on mortality convergence across industrialized countries. *Population and Development Review*, 31(4):645–674.

Gampe, J. (2021). Mortality of supercentenarians: Estimates from the updated IDL. In Maier, H., Jeune, B., and Vaupel, J. W., editors, *Exceptional Lifespans*, pages 29–35. Springer International Publishing.

Human Mortality Database (2023). University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at `www.mortality.org` or `www.humanmortality.de`. Data retrieved on March 28, 2023.

Janssen, F. and de Beer, J. (2019). The timing of the transition from mortality compression to mortality delay in Europe, Japan and the United States. *Genus*, 75:1–23.

Kannisto, V. (2001). Mode and dispersion of the length of life. *Population: An English Selection*, pages 159–171.

Kannisto, V., Lauritsen, J., Thatcher, A. R., and Vaupel, J. W. (1994). Reductions in mortality at advanced ages: several decades of evidence from 27 countries. *Population and Development Review*, pages 793–810.

Keyfitz, N. (1977). What difference would it make if cancer were eradicated? An examination of the Taeuber paradox. *Demography*, 14:411–418.

Leser, C. E. V. (1955). Variations in mortality and life expectation. *Population Studies*, 9(1):67–71.

Newman, S. J. (2018). Errors as a primary cause of late-life mortality deceleration and plateaus. *PLOS Biology*, 16(12):e2006776.

Permanyer, I. and Shi, J. (2022). Normalized lifespan inequality: disentangling the longevity–lifespan variability nexus. *Genus*, 78(1):1–29.

R Core Team (2022). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Riffe, T., van Raalte, A., and Dudel, C. (2023). *LifeIneq: An R-package to calculate measures of lifespan variation from a life table*. Available at `https://github.com/alysonvanraalte/LifeIneq`.

Schmertmann, C. P. (2020). Revivorship and life lost to mortality. *Demographic Research*, S8(17):497–512.

Ullrich, N., Schmertmann, C., and Rau, R. (2022). Death Expectancy H_1: An Alternative Measure for Longevity. In *PAA 2022 Annual Meeting*. Atlanta, Georgia & online.

Van Raalte, A. A. and Caswell, H. (2013). Perturbation analysis of indices of lifespan variability. *Demography*, 50(5):1615–1640.

Vaupel, J. W. and Canudas-Romo, V. (2003). Decomposing change in life expectancy: A bouquet of formulas in honor of Nathan Keyfitz's 90th birthday. *Demography*, 40(2):201–216.

Vaupel, J. W., Carey, J. R., Christensen, K., Johnson, T. E., Yashin, A. I., Holm, N. V., Iachine, I. A., Kannisto, V., Khazaeli, A. A., Liedo, P., et al. (1998). Biodemographic trajectories of longevity. *Science*, 280(5365):855–860.

Wilmoth, J. R., Andreev, K., Jdanov, D., Glei, D. A., Riffe, T., Boe, C., Bubenheim, M., Philipov, D., Shkolnikov, V., Vachon, P., Winat, C., and Barbieri, M. (2021). Methods Protocol for the Human Mortality Database. Last Revised: January 26, 2021 (Version 6).

Wilmoth, J. R. and Horiuchi, S. (1999). Rectangularization revisited: Variability of age at death within human populations. *Demography*, 36(4):475–495.

Wood, S. (2017). *Generalized Additive Models: An Introduction with R*. Chapman and Hall/CRC, 2 edition.

Zhang, Z. and Li, Q. (2020). The ratio of expansion to compression: A new measure of lifespan disparity. *PLOS One*, 15(12):e0243482.